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# 2.0 Time and Frequency

OBJECTIVES

This section will endeavor to:

- Consider the relationship between frequency and time
- Examine applications of the Fourier Transform
- Observe the difference between continuos and discrete functions

ADDITIONAL RESOURCES

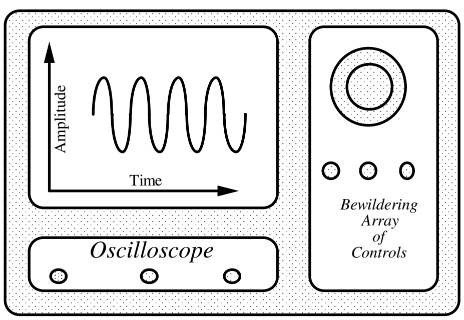
Fourier analysis and sampling:

Signals can be analyzed mathematically [boring], by test equipment [fun], or by means of computer simulation [wow]. Inevitably, some may wonder: why bother with the math and just use the equipment? The answer is actually quite simple: the math is cheaper and more versatile. Besides that, if one doesn't understand the math, it is impossible to understand the simulation. As for the equipment, it's terribly expensive.

Test equipment operates on signals, but mathematics applies to both signals and systems. With few exceptions, it is not practical to design a system and then test it to see if it works. Rather, the system must be designed, analyzed and simulated to determine its viability before significant amounts of money are invested on its construction.

## 2.1 Characterizing Signals

One of the first instruments a technologist learns to use is the oscilloscope.

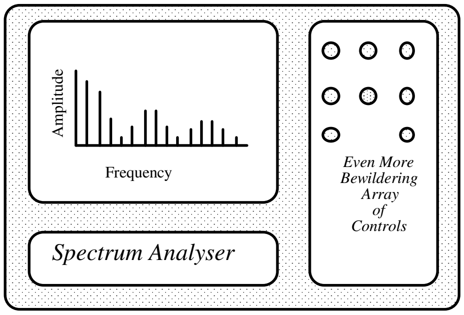



This device creates a visual representation of electrical signals in the time domain, and opens a window into the otherwise hidden world of electricity. Its basic controls adjust the display of time and amplitude attributes. Most oscilloscopes have two channels, which allows two signals to be compared.

A logic analyzer is an oscilloscope of sorts, but is used to represent digital signals in the time domain. Generally, it ignores or limits examination of amplitude variations, and concentrates on timing concerns. Most logic analyzers have 8 or more display channels.

A little later, the technologist learns to use the spectrum analyzer, a device that creates a visual representation of electrical signals in the frequency domain. Its basic controls adjust the display of frequency and amplitude attributes.

http://www.picotech.com/applications/ampdesign.html




There are all kinds of electronic signals: random, periodic, aperiodic, analog, digital etc. Each requires a different analytical approach, and different test equipment.

Random signals have no discernible pattern and are not predictable. Periodic signals are repetitive and have a finite period. Aperiodic signals are somewhere in between.

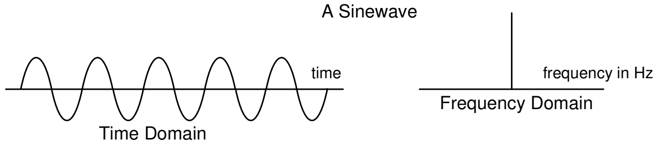
Analog signals can have any amplitude value, but digital signals have a finite set of amplitude values. The most common digital signal consists of two states and is known as a binary signal.

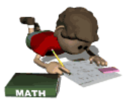
To start our journey into signals, we will examine a sinewave, a square wave, and then a single binary pulse.

The mathematical tool that helps us understand signals and relates the time and frequency domains is the not so simple Fourier Transform.

## 2.1.1 A Sinewave

The sine function is a basic trigonometric function, and its characteristics are well understood. It is used to describe trigonometric relationships and harmonic motion.

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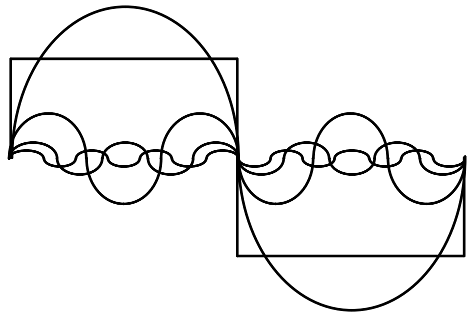
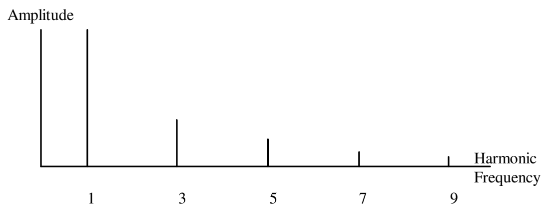
It is interesting to note, that if many sinewaves are added together, new shapes can be formed. In fact, it is possible to create any periodic waveform by adding enough sinewaves of the right amplitude and frequency.

### 2.1.2 A Square Wave

A square wave can be composed of a number of sinewaves, or harmonic components. The more harmonic components used, the better the square wave.



Since each one of the sinewave components has a unique frequency domain signature, we can sketch the frequency domain of the square wave as:

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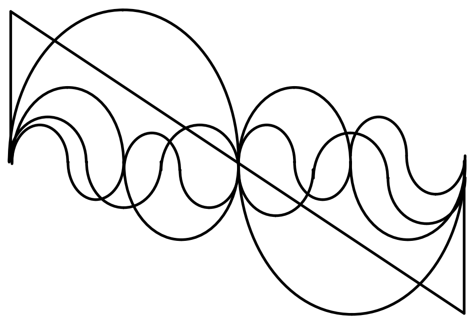
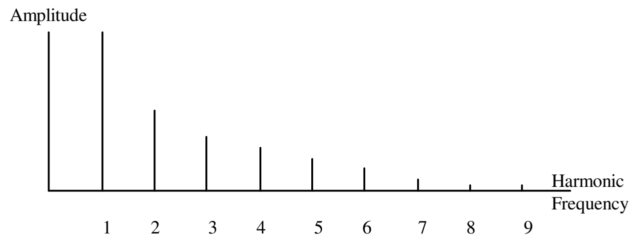
### 2.1.3 A Triangle Wave

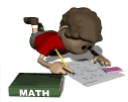
A triangle wave can likewise be composed of a number of sinewaves, or harmonic components. Again, the more harmonic components used, the better the triangle wave appears.



Since each one of the triangle wave components has a unique frequency domain signature, we can sketch the frequency domain of the triangle wave as:

_____ Notes _____




SystemView BY ELANIX 

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## 2.2 Periodic Signals and the Discrete Fourier Transform

A periodic signal is one that continuously repeats a given pattern. Such signals can be decomposed into a series of harmonically related sinewaves by using the Fourier Transform. The resulting display on a spectrum analyzer is a discrete or picket fence spectrum.

The Fourier transform is given by:

$$\mathrm{F}\{f(t)\} \equiv F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

Although this expression appears quite complex, it is relatively easy to apply when analyzing simple time domain waveforms. Complex waveforms however, are not quite so easy to analyze. If you are interested in the math, check out Appendix 3.

The exponential term in the Fourier transform can be interpreted as a rotating vector. This is perhaps more readily seen by using Euler's identity:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

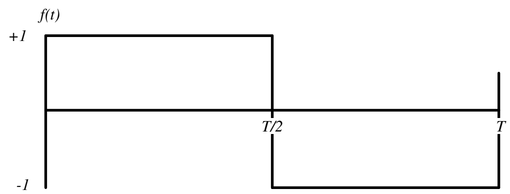
Plotting this function in a complex plane generates a unit circle.

In the following equations, it is generally understood that:

$$\omega = 2\pi f \qquad f = \frac{1}{T}$$

## 2.2.1 Fourier Expansions of Periodic Functions

SQUARE WAVE



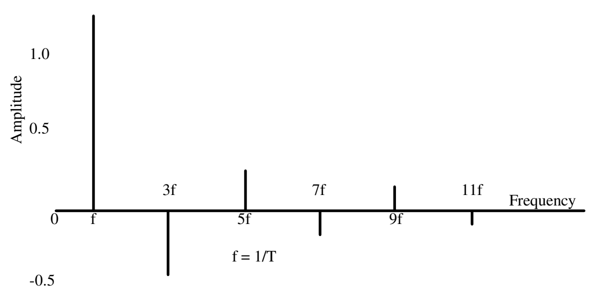

A square wave consists of an infinite series of odd numbered harmonics:

$$f(t) = \frac{4}{\pi}\left[\cos(\omega t) - \frac{1}{3}\cos(3\omega t) + \frac{1}{5}\cos(5\omega t) - \cdots\right]$$

This expression can also be written in the closed form [i.e. without the dots...]:

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{2\pi nt}{T}\right)$$

A spectral plot of these components resembles:




Each vertical line represents a discrete frequency. However, since a spectrum analyzer measures the absolute magnitude of a signal, its display (in a linear, not dB mode) would resemble:

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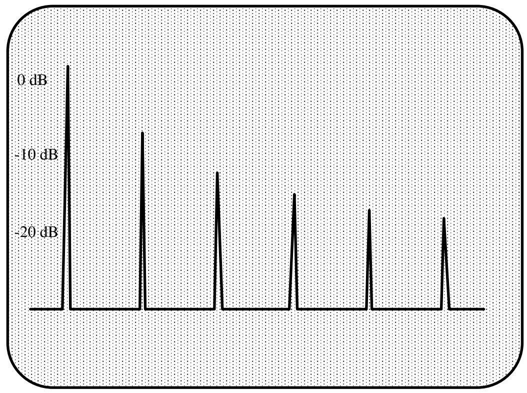
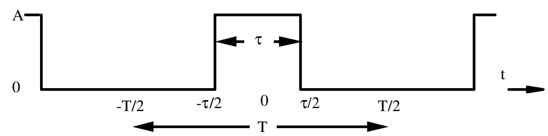
Since the amplitude drops off quickly as the harmonic number increases, the spectrum analyzer often is used in the dB mode, in which case the display would resemble:




### 2.2.1.1 A DC Pulse Series



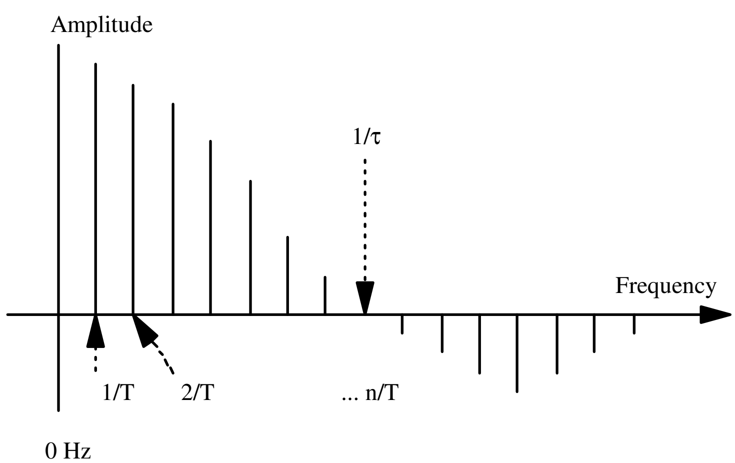

Depending on the duty cycle, this waveform consists of both even and odd numbered harmonics as well as a dc term:

$$f(t)=\frac{A\tau}{T}+\frac{2A}{\pi}\left[\sin\left(\frac{\omega\tau}{2}\right)\cos(\omega t)+\frac{1}{2}\sin\left(\frac{2\omega\tau}{2}\right)\cos(2\omega t)+\frac{1}{3}\sin\left(\frac{3\omega\tau}{2}\right)\cos(3\omega t)+\cdots\right]$$

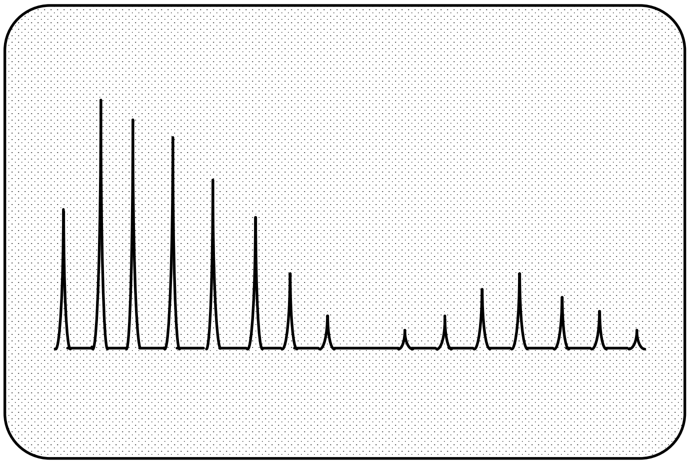
or in the closed form:

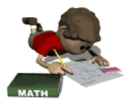
$$f(t)=\frac{A\tau}{T}+\sum_{n=1}^{\infty}\frac{2A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)$$

A plot of this resembles:



On a spectrum analyzer, this signal would be displayed as:



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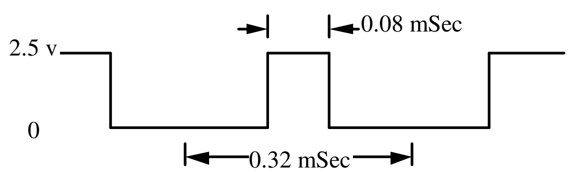
EXAMPLE CALCULATION

It may be instructive to work through an example and see how to calculate the frequency and amplitude of the harmonic components. For instance, lets find the frequency content of the following signal displayed on an oscilloscope:




SystemView BY ELANIX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We are going to use the formula:

$$f(t)=\frac{A\tau}{T}+\frac{2A}{\pi}\left[\sin\left(\frac{\omega\tau}{2}\right)\cos(\omega t)+\frac{1}{2}\sin\left(\frac{2\omega\tau}{2}\right)\cos(2\omega t)+\frac{1}{3}\sin\left(\frac{3\omega\tau}{2}\right)\cos(3\omega t)+\cdots\right]$$

The average value or dc component [0 Hz] is:

$$a_o=\frac{A\tau}{T}=\frac{2.5\times 0.08\times 10^{-3}}{0.32\times 10^{-3}}=0.625 \quad \text{Volts}$$

The first harmonic or fundamental frequency is:

$$f = \frac{1}{T} = \frac{1}{0.32 \times 10^{-3}} = 3.125 \text{ KHz}$$

Each harmonic is an integer multiple of this frequency.

The amplitude of the fundamental frequency is:

$$a_1 = \frac{2A}{\pi}\sin\left(\frac{\omega\tau}{2}\right) = \frac{2\times 2.5}{\pi}\sin\left(\frac{2\times\pi\times 3.125\times 10^3\times 0.08\times 10^{-3}}{2}\right)$$
$$= 1.125395395$$

The amplitude of the 2nd harmonic is:

$$a_2 = \frac{2A}{2\pi}\sin\left(\frac{2\omega\tau}{2}\right) = 0.795774715459$$

The amplitude of the 3rd harmonic is:

$$a_3 = \frac{2A}{3\pi}\sin\left(\frac{3\omega\tau}{2}\right) = 0.375131798399$$

This process can be carried on indefinitely and we can calculate the amplitude and frequency of all of the spectral components.

The duty cycle is a fractional measure of the pulse width.

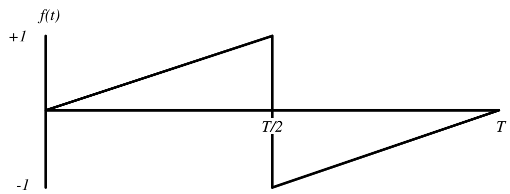
$$\text{Duty Cycle} = \frac{\tau}{T} = \frac{0.08\text{mSec}}{0.32\text{mSec}} = \frac{1}{4} \text{ or } 25\%$$

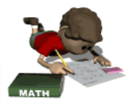
| Harmonic | Amplitude [Volts] $a_n$ | Frequency [KHz] $nf$ |
|---|---|---|
| 0 | 0.625 | 0 |
| 1 | 1.125395395 | 3.125 |
| 2 | 0.795774716 | 6.25 |
| 3 | 0.375131798 | 9.375 |
| 4 | 0 | 12.5 |
| 5 | -0.22507907 | 15.625 |
| 6 | -0.26525823 | 18.75 |
| 7 | -0.16077077 | 21.875 |
| 8 | 0 | 25 |
| 9 | 0.125043933 | 28.125 |
| 10 | 0.159154943 | 31.25 |

Note that in this example, the amplitude of every 4th harmonic is zero, and the duty cycle is 1/4. Likewise a pulse with a 1/2 or 50% duty cycle is comprised of only odd harmonics, since every second harmonic is zero.

Ramp and triangular waveforms are also quite easy to decompose into harmonic components.

### 2.2.1.2 Ramp Waveform




 HTML

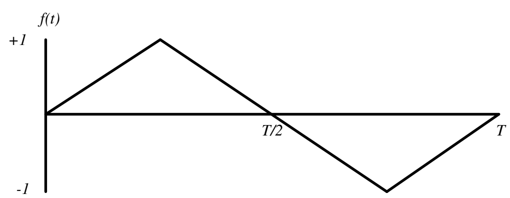
This waveform consists of an infinite series of harmonics:

$$f(t)=\frac{2}{\pi}\left[\sin(\omega t)-\frac{1}{2}\sin(2\omega t)+\frac{1}{3}\sin(3\omega t)-\cdots\right]$$

or in the closed form:

$$f(t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{2\pi nt}{T}\right)$$

### 2.2.1.3 A Triangle Waveform

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This waveform consists of a series of odd harmonics:

$$f(t)=\frac{8}{\pi^2}\left[\sin(\omega t)-\frac{1}{3^2}\sin(3\omega t)+\frac{1}{5^2}\sin(5\omega t)-\cdots\right]$$

or in the closed form:

$$f(t)=\frac{8}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2}\sin\left(\frac{2\pi nt}{T}\right)$$

Time domain signals and their frequency domain equivalents form what is known as a **Fourier Transform pair**.

## 2.3 Sampling Function

The sampling process occurs in analog to digital converters and will be covered in more detail in section 3. One way to regard this process is as a form of multiplication.

The multiplication of two sinewaves results in two new frequencies being created. The new frequencies are the sum and difference of the originals. Notice how this is shown by the following trigonometric identity:

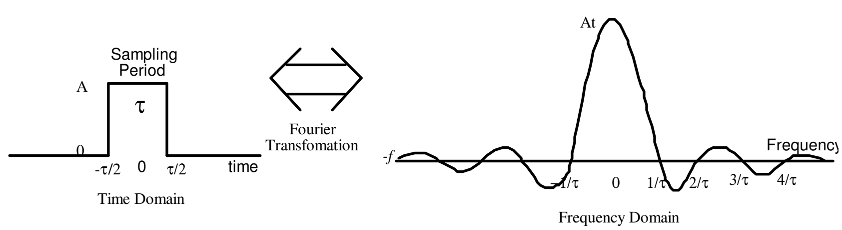
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha - \beta) - \frac{1}{2}\cos(\alpha + \beta)$$

From this we note that the resulting frequencies have half the amplitude of the originals. The change from sin to cosine simply corresponds to a phase change of 90°, and the negative sign corresponds to an additional phase change of 180°. Substituting in two frequencies we obtain:

$$\sin(\omega_1 t)\sin(\omega_2 t) = \underbrace{\frac{1}{2}\cos(\omega_1 - \omega_2)t}_{\text{difference frequency}} - \underbrace{\frac{1}{2}\cos(\omega_1 + \omega_2)t}_{\text{sum frequency}}$$

From this we observe that if we multiply a 10 KHz sine wave by a 2 KHz sine wave, the result is two sine waves at 8 and 12 KHz. This multiplication phenomenon, which results in frequency shifting, is also known as heterodyning. It is used in all radio transmitters and receivers to create IF or intermediate frequencies.

A sampling signal is simply a pulse, the frequency response of which is given by the Fourier Transform. It is interesting to note that if a series of sampling pulses are used, the result as we have already seen is a discrete or picket fence spectrum. If however, only one pulse is used, it has a continuous or smooth spectrum as follows:

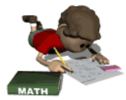



These two sketches form a Fourier Transform Pair. The continuous frequency domain function is of the form $\frac{\sin x}{x}$ and occurs frequently in signal analysis. It is simply called the **sampling function**. It will pre-distort the frequency content of any sampled analog signal, by creating additional sum and difference frequencies. This means that a sampled signal will not have the same spectral content of the original.

If the sampling signal has a dc or zero frequency component, one of the results of the multiplication or sampling process, will be the original sampled signal itself. However, to complicate matters even further, the sampling process may also alter the frequency content of the original signal. This phenomenon is sometimes referred to as windowing. Applications such as audio CDs require complex digital signal processing circuits to minimize this affect.

The sampling function is defined by:

$$\mathrm{Sa}(x) = \frac{\sin x}{x} \quad \text{and} \quad \mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

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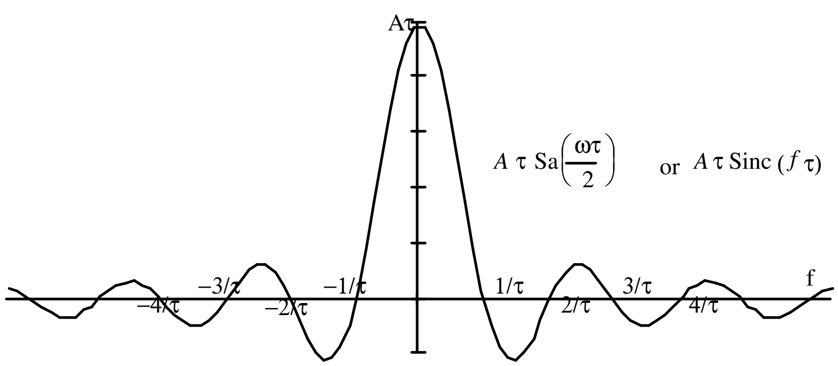
In the frequency domain it resembles:





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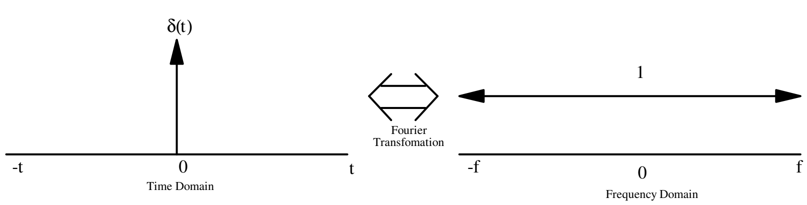
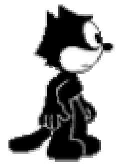
The ideal sampling signal is infinitely thin. Since the Fourier Transform of this signal shows that it contains all possible frequencies, it cannot add distortion during the sampling process. Many textbooks use this type of signal as the sampling waveform to eliminate windowing effects.

This theoretical signal is known as the **Dirac Delta Pulse**:




By definition, the delta pulse has infinite amplitude, zero width, an area of one, and its spectrum contains all frequencies.

Notes



# Assignment Questions

## Quick Quiz

1. The expression $Sa(x) = \frac{\sin(x)}{x}$ occurs frequently in signal analysis and is called the [sampling, Nyquest, Fourier] function.
2. The frequency spectrum of a 50% duty cycle square wave contains only [even, odd] harmonics.
3 The unit impulse response [can, cannot] be used to completely describe the dynamic behavior of a linear system.
4. The ideal sampling pulse contains [some, all, no] frequencies.
5. [Random, Periodic] signals have well defined spectra.
6. It is impossible to create any periodic waveform by adding enough sinewaves of the right amplitude and frequency. [True, False]
7. Plotting Euler's Identity in a complex plane produces a [sinewave, square wave, circle, triangle].
8. Not all periodic waveforms have a fundamental frequency component. [True, False]
9. Sampling a waveform is a form of time domain [addition, subtraction, multiplication].
10. Multiplication in the time domain creates [addition & subtraction, multiplication & division] in the frequency domain.
11. The delta pulse [is, is not] a theoretical convenience.

## Analytical Problems

1. Make a sketch of three different signals having:

   a) No symmetry under any circumstance

   b) Only odd symmetry

   c) Only even symmetry

2. Tabulate the amplitude and frequency for the first 10 harmonics of a pulse train with the following characteristics:

   Amplitude = 1.1 volts

   Pulse period = 0.6 μSec

   Pulse duration = 0.1 μSec

## Composition Questions

1. What is the difference in the frequency spectrum of a single pulse and a series of pulses?
2. What is the difference between the open and closed forms of a mathematical expression?
3. What is the effect of removing the fundamental frequency component from a periodic waveform?
4. Why do spectrum analyzers have linear and dB display modes?
5. What is windowing?

## SystemView Simulations

1. Plot the spectrum of a 1 Hz pulse with a 50% and 10% duty cycle.
2 Plot the spectrum of a 1 Hz ramp.
3. Plot the spectrum of a delta pulse.

Notes

# For Further Research

Bissell C C & Chapman D A, *Digital Signal Transmission*, Cambridge University Press, 1992

Bracewell Ronald N, *The Fourier Transform and Its Applications* [2nd ed.], McGraw-Hill 1978

Gupta, Bayless, & Peikari, *Circuit Analysis*,

Hayt & Kemmerly, *Engineering Circuit Analysis*,

Johnson, Hilburn, & Johnson, *Basic Electric Circuit Analysis*,

Kreyszig Erwin, *Advanced Engineering Mathematics* [6th ed.], John Wiley & Sons, 1988

Murdock Joseph, *Network Theory*,

Poularikas & Seely, *Signals & Systems*, PWS Publishers, 1985

Scott, *Linear Circuits*,

Van Valkenburg, *Network Analysis*,

Wing Omar, *Circuit Theory,*

CRC Standard Mathematical Tables

Fourier Transforms

http://www.spd.eee.strath.ac.uk/~interact/FFT/fourier.html

http://www.amara.com/IEEEwave/IW_fourier_ana.html

http://svr-www.eng.cam.ac.uk/~ajr/SA95/

http://www.spd.eee.strath.ac.uk/~interact/fourier/

http://aurora.phys.utk.edu/~forrest/papers/fourier/index.html

http://www.astro.virginia.edu/~eww6n/math/FastFourierTransform.html

http://www.intersrv.com/~dcross/fft.html

http://capella.dur.ac.uk/doug/fourier.html