Appendix 2, LDEs

First Order Linear Differential Equations

Any passive electronic circuit containing an inductor or capacitor can be mathematically described as a first order LDE. The general form of this equation is:

$$\frac{dx}{dt} + Px = Q \tag{1}$$

Where P is a constant, and Q may be a function of the independent variable t or a constant. The inevitable question arises: What is x? In electronic circuits is most often a voltage or current. To determine how these operate in a circuit, it is necessary to solve these basic equations in terms of the dummy variable x. The general solution to this equation is:

$$x = \underbrace{e^{-Pt}\int Qe^{Pt}dt}_{\text{particular integral}} + \underbrace{Ke^{Pt}}_{\text{complementary factor}} \tag{2}$$

For any network, P will be a positive constant determined by the network components and Q will be either a forcing function or its derivative.

We well might wonder at how this solution was obtained. The term which causes the trouble is $\frac{dx}{dt}$ Notice that a solution cannot be obtained by simply integrating the equation and solving for x since we'd be left with the question: What is the integral of x? Therefor we must modify the expression is such a way that when the integral is taken, the dummy variable x is left on its own.

To do this, the general equation (1) is multiplied by an integrating factor e^{Pt}. If P were a function of time, the proper integrating factor would be $e^{\int Pdt}$. In any case, the result is:

$$\underbrace{e^{Pt}\frac{dx}{dt} + Pxe^{Pt}}_{\text{find an equivalent expression for this term}} = Qe^{Pt} \tag{3}$$

Recall that the derivative of a product is given by:

$$d(xy) = xdy + ydx \tag{4}$$

$$\begin{array}{llll} \text{let} & x = x & \text{and} & y = e^{Pt} \\ \text{then} & dx = dx & \text{and} & dy = Pe^{Pt} \end{array} \tag{5}$$

Substituting equation 5 into 4, we obtain:

$$d\left(xe^{Pt}\right) = xPe^{Pt} + e^{Pt}dx \tag{6}$$

Note that the RHS of Equation 6 is identical to the LHS of equation 3. Substituting this back into equation 3, we obtain:

$$d\left(xe^{Pt}\right) = Qe^{Pt} \tag{7}$$

Integrating both sides with respect to time, we obtain:

$$\begin{aligned} xe^{Pt} &= \int Qe^{Pt}dt + K \\ \therefore X &= e^{Pt}\int Qe^{Pt}dt + e^{Pt}K \end{aligned} \tag{8}$$

which amazingly enough is identical to equation 2.

Impulse Function

The unit impulse is one of the most useful theoretical signal types in common use today. Its designation is δ(t), is infinitely thin and has by definition, unity area.

Some of its useful properties are:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \text{or} \qquad \int_{-0}^{+0} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0) \qquad \text{or} \qquad \int_{-0}^{+0} f(t)\delta(t-t_0)dt = f(t_0)$$