# Appendix 3, Fourier Transforms

In the 1750's, Bernoulli suggested that physical motion could be represented by a linear combination of harmonics. In 1822, Fourier proved this true.

The Fourier Transform is used to switch between oscilloscope time domain representations, and spectrum analyzer frequency domain representations. What is so remarkable about this mathematical tool is that it was invented a long time before these electronic devices were even conceived.

The transform is used to determine the frequency content of signals.

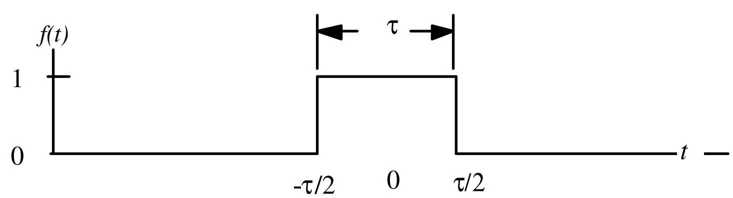
### The Fourier Transform:

$$\mathrm{F}\{f(t)\} \equiv F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

In mathematical terms, the Fourier transform converts a piecewise continuous periodic time domain signals into the frequency domain. The inverse transform performs the reverse operation.

Most signals are described in the time domain, but the channels or systems used to convey them are generally characterized in the frequency domain. It is therefore quite useful to determine in advance whether a given signal will fit into a communications channel or how much it will be altered by passing through it.

### How to describe a binary pulse




This is function can be written as: $f(t) = \begin{cases} 1 \frac{-\tau}{2} \le t \le \frac{\tau}{2} \\ 0 \text{elsewhere} \end{cases}$

and is well defined in three separate regions:

$$f(t) = 0 \quad t < \frac{-\tau}{2}$$

$$f(t) = 1 \quad \frac{-\tau}{2} \le t \le \frac{\tau}{2}$$

$$f(t) = 0 \quad t > \frac{-\tau}{2}$$

Notes

Note: Some authors prefer the notation:

$$\prod\left(\frac{t}{\tau}\right) \equiv \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}$$

Taking the Fourier transform of this function requires three integrals, one for each defined value of *f(t)*.

$$\mathrm{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\frac{\tau}{2}} 0e^{-j\omega t}\,dt + \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} 1e^{-j\omega t}\,dt + \int_{\frac{-\tau}{2}}^{\infty} 0e^{-j\omega t}\,dt$$

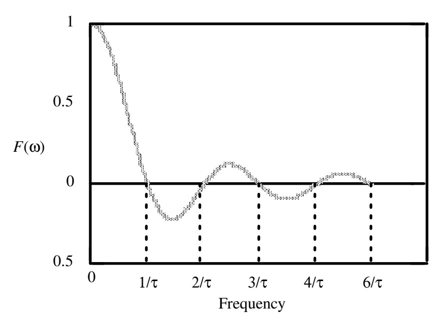
The value of the first and third integrals is zero since any function times zero is zero, and we are left with:

$$F(\omega) = \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} 1e^{-j\omega t}\,dt$$

$$= \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} = -\frac{1}{j\omega}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right] = \frac{2}{\omega}\left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right]$$

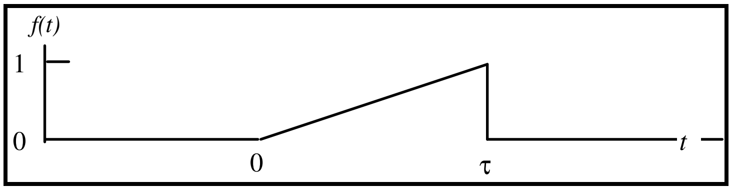
$$= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

Notice that this function has only real components. A plot of this function was created using MathCAD:




From this we observe that a pulse contains a wide range of frequencies. However, their significance decreases as frequency increases. It is also interesting to note that all integer multiples of 1/τ = 0.

Let's take the Fourier Transform of another pulse shape.



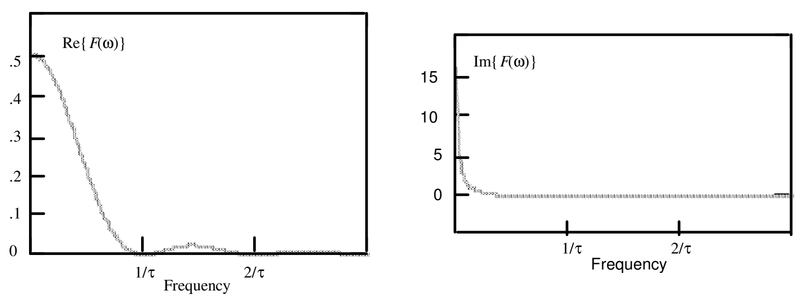

The first thing to do is to find an expression that describes the function *f(t)*. In this case it is quite easy since *f(t)* is simply a straight line between the limits of 0 and τ.

$$f(t) = \frac{1}{\tau} t \qquad\qquad 0 \le t \le \tau$$

Now we can apply the Fourier transform:

$$F(\omega) = \int_0^{\tau} \frac{1}{\tau} t e^{-j\omega t} dt = \frac{1}{\tau}\left[\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_0^{\tau} = -\frac{2}{\tau\omega^2}\left[e^{-j\omega\tau} - 1\right]$$
$$= \frac{2}{\tau\omega^2}\left[1 + e^{-j\omega\tau}\right]$$

Note that this function has real and imaginary components. Plots of these components are:




In this case, the real part represents the ramp pulse spectrum.

## Symmetry

By using Euler's identity:

$$e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$$

The Fourier transform can be written as:

$$\mathrm{F}\{f(t)\} \equiv F(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

If *f(t)* exhibits symmetry, the analysis is simplified. For example, if *f(t)* is an even function (like Cosine), the Fourier transform reduces to:

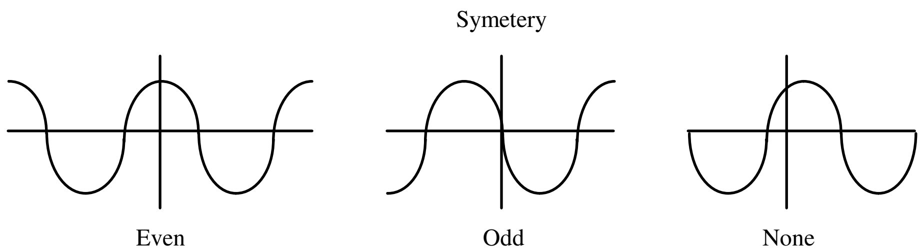
$$\mathrm{F}\{f(t)_{even}\} \equiv F(\omega)_{real} = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt$$

If $f(t)$ is an odd function (like Sine), the transform reduces to:

$$\mathrm{F}\{f(t)_{odd}\} \equiv F(\omega)_{imag} = -j\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$$

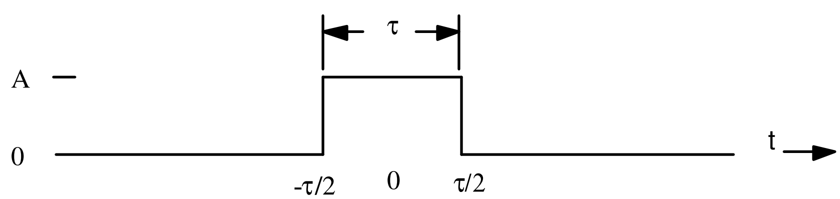
If $f(t)$ has no symmetry, then both of these transforms must be used. The situation becomes even more complex if $f(t)$ is itself a complex function. In that case, there may be complex transform pairs generated.

The same waveform can sometimes be described differently simply by placing the origin of the time-amplitude axis at a different place.




In this example the signal can be correctly described as having even, or odd symmetry. Since selecting an arbitrary time-amplitude origin almost invariably produces no symmetry, the category *no symmetry* is reserved for signals which do not manifest symmetry under any circumstance. Some signals will manifest eveness but not oddness and vice versa.

### Using Symmetry to find the Fourier Transform of a Pulse




By observing the symmetry we can use the formula:

$$\mathrm{F}\{f(t)_{even}\} \equiv F(\omega)_{real} = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt$$

We do not have to evaluate the integral during the period when $-\tau/2 > t$ or $t > \tau/2$ since during those intervals $f(t) = 0$, and the integral will also equal zero.

$$F(\omega) = \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} A\cos(\omega t)\,dt = A\frac{\sin(\omega\tau)}{\omega}\Bigg|_{\frac{-\tau}{2}}^{\frac{\tau}{2}} == \frac{A}{\omega}\left[\sin\left(\frac{\omega\tau}{2}\right) - \sin\left(\frac{-\omega\tau}{2}\right)\right]$$

$$= \frac{2A}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

We arrive at the same solution but more quickly than before.

Notes

## Periodic Signals and the Discrete Transform

The Fourier Transform of a single pulse results in a continuous transform having a continuous spectrum. This simply means that the plot results in a smooth curve. The Fourier transform of a series of pulses results in a discrete transform, which has a 'picket fence' or discrete spectrum.

A periodic signal is one that continuously repeats a given pattern. Such signals can be decomposed into a linear combination of harmonics by the following trigonometric series:

$$f(x) = \underbrace{a_0}_{\text{a constant}} + \sum_{n=1}^{\infty}\left[\underbrace{a_n \cos(nx)}_{\text{even harmonics}} + \underbrace{b_n \sin(nx)}_{\text{odd harmonics}}\right]$$

In the above equation, the terms even and odd refer to symmetry, not the even and odd multiple of the fundamental frequency component.

The problem with this expression is finding the coefficients $a_o$, $a_n$, and $b_n$. These coefficients can be determined by integration. If we define function period as $2\pi$, then the integration limits are $\pm\pi$.

To find $a_0$ , integrate both sides of the expression:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \underbrace{\sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n \cos(nx)\,dx + \int_{-\pi}^{\pi} b_n \sin(nx)\,dx\right]}_{=0}$$

$$= 2\pi a_0$$

Solving for $a_o$ we obtain:

$$\therefore \quad a_0 = \frac{1}{2\pi}\int (x)\,dx$$

This term corresponds to an average value since it is simply the area under a curve divided by the period ($2\pi$).

To find $a_n$ , multiply both sides of the equation by $\cos(mx)$ and integrate again.

$$\int_{-\pi}^{\pi} f(t)\cos(mx)dt = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]\right]\cos(mx)dx$$

$$= a_0 \underbrace{\int_{-\pi}^{\pi} \cos(mx)dx}_{=0}$$

$$+ a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx}_{\substack{=0\,for\,n\neq m \\ =\pi\,for\,n=m}}$$

$$+ b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx}_{=0}$$

letting $n = m$:

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = a_n\pi$$

Therefor:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

Similarly, by multiplying both sides by sin($mx$) and integrating once more, we obtain:

$$b_n = \frac{1}{\pi}\int \quad (x)\sin(nx)\,dx$$

By changing the dummy variables in the above equations, we can convert them to something more familiar:

$$\text{let} \quad x = \frac{2\pi t}{T} \qquad \text{then} \quad dx = \frac{2\pi}{T}\,dt$$

If $T$ is the waveform period, we obtain:

$$\mathrm{F}\{f(t)\} \equiv F(\omega) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right)\right]$$

where

$$a_o = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)dt$$

$$a_n = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

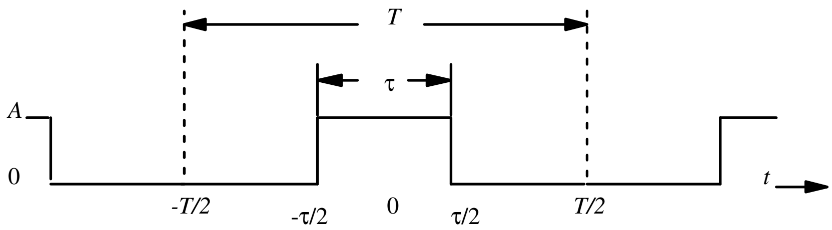
This is the harmonic version of the Fourier Transform and forms an extremely useful set of equations.

## Using the Harmonic Fourier Transform

The harmonic transform is also called the discreet transform, since it results in specific values.

### Spectrum of a Pulse Train

A signal composed of a series of identical pulses resembles:




As in the single pulse case, there are three distinct intervals during the waveform period *T*. Since the regions between -*T/2* to -τ/2 and τ/2 to *T/2* are equal to zero, we can ignore them. The problem therefor reduces itself to the area between -τ/2 and τ/2.

Since the waveform as drawn has even symmetry the coefficient $b_n = 0$.

$$a_0 = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\,dt$$

$$= \frac{1}{T} At\Big|_{-\tau/2}^{\tau/2} = \frac{1}{T}\left[A\frac{\tau}{2} - \left(-A\frac{\tau}{2}\right)\right]$$

$$= \frac{A\tau}{T}$$

$$a_n = \frac{2}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$= \frac{2A}{T}\left[\frac{T}{2\pi n}\sin\left(\frac{2\pi nt}{T}\right)\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{A}{\pi n}\left[\sin\left(\frac{\pi n\tau}{T}\right) - \sin\left(\frac{-\pi n\tau}{T}\right)\right]$$

$$= \frac{2A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right)$$

$$b_n = \frac{2}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$= \frac{2A}{T}\left[-\frac{T}{2\pi n}\cos\left(\frac{2\pi nt}{T}\right)\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = -\frac{A}{\pi n}\left[\cos\left(\frac{\pi n\tau}{T}\right) - \cos\left(\frac{-\pi n\tau}{T}\right)\right]$$

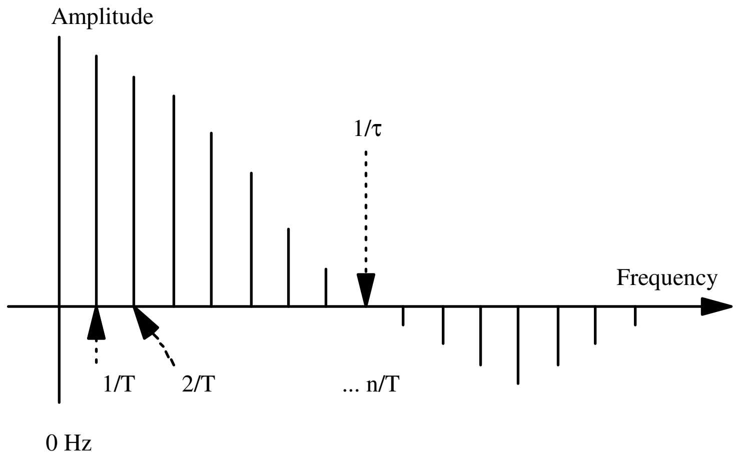
$$= 0$$

Putting it all together, the spectrum of the above pulse train is given by:

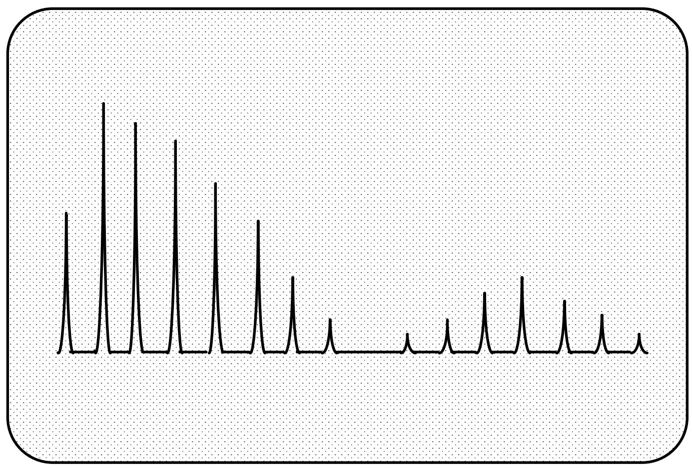
$$\mathrm{F}\{f(t)\} = \frac{A\tau}{T} + \sum_{n=1}^{\infty}\frac{2A}{\pi n}\sin\left(\frac{\pi n\tau}{T}\right)\cos\left(\frac{2\pi nt}{T}\right)$$

A plot of which resembles:

Notes

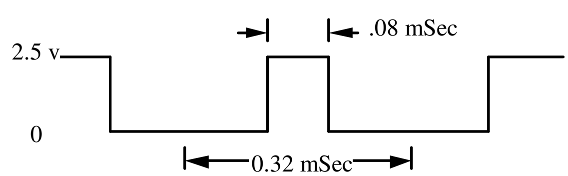



Note that instead of a continuous function, we obtain a discrete one. Each of these values represents a discrete frequency. On a spectrum analyzer, this display would appear as:



Notice that the spectrum analyzer does not display any negative amplitude values. This is because a negative amplitude in an ac signal simply means that there is a 180$^o$ phase inversion. The analyzer does not display signal phase, but only magnitude and frequency.

It may be instructive to work through an example. For instance, lets find the frequency content of the following signal as displayed on an oscilloscope:




The average value or dc component [0 Hz] is:

$$a_o = \frac{A\tau}{T} = \frac{2.5 \times 0.08 \times 10^{-3}}{0.32 \times 10^{-3}} = 0.625 \quad \text{Volts}$$

Notes

The fundamental frequency is:

$$f = \frac{1}{T} = \frac{1}{0.32 \times 10^{-3}} = 3.125 \text{ KHz}$$

The amplitude of the harmonic frequency components is:

$$a_n = \frac{2A}{n\pi} \sin\left(\frac{n\pi\tau}{T}\right) = \frac{2 \times 2.5}{n\pi} \sin\left(\frac{n\pi \times 0.08 \times 10^{-3}}{0.32 \times 10^{-3}}\right)$$
$$= \frac{1.59154931}{n} \sin(0.785398163\ n)$$

From this we can calculate the amplitude and frequency of all of the spectral components:

| Harmonic | Amplitude [Volts] $a_n$ | Frequency [KHz] $nf$ |
|---|---|---|
| 0 | 0.625 | 0 |
| 1 | 1.125395395 | 3.125 |
| 2 | 0.795774716 | 6.25 |
| 3 | 0.375131798 | 9.375 |
| 4 | 0 | 12.5 |
| 5 | -0.22507907 | 15.625 |
| 6 | -0.26525823 | 18.75 |
| 7 | -0.16077077 | 21.875 |
| 8 | 0 | 25 |
| 9 | 0.125043933 | 28.125 |
| 10 | 0.159154943 | 31.25 |

It is interesting to note that the amplitude of every 4th harmonic in this example is zero. This also happens to be the duty cycle of the pulse. It is easy to show that a pulse with 50% duty cycle will be comprised of only odd harmonics, since every second one will equal zero.

## Characterizing Systems in the Time Domain

Just as it is useful to transform time domain functions into the frequency domain, it is often just as useful to do the reverse. Although transmission systems are most often specified in terms of the frequency domain, they can be analyzed quite effectively in the time domain.

### Dirac Delta Pulse

In examining spectrum of a plat topped pulse, we may well wonder, what would happen if the pulse were very narrow.

If we keep the pulse area constant, [i.e. $A\tau = 1$] while decreasing the pulse width [$\tau$], the frequency domain envelope expands outward while the envelop peak remains constant at 1.

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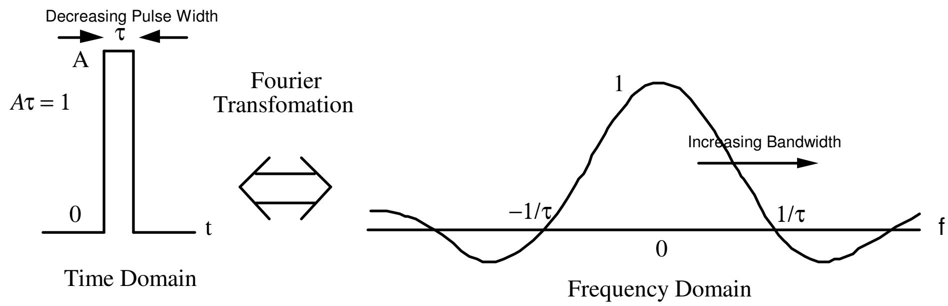
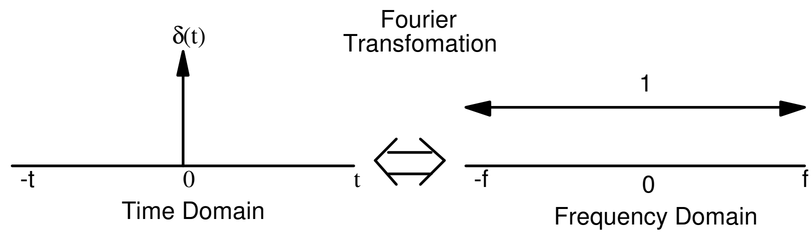



From this we observe that the transform of a unity area zero width pulse, is a flat line in the frequency domain.

Graphically, the Fourier pair is represented by:




This means that a single infinitely narrow pulse actually contains all frequencies for the short duration of its existence. The unit impulse is often drawn with an arrow to indicate that the magnitude is undefined, but it should be remembered that its area is always equal to 1.

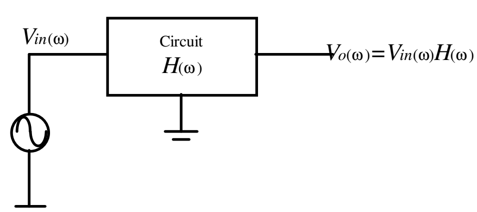
This transform pair illustrates why sharp transients created by lightening, ignition systems, or motor brushes, creates wideband radio frequency interference.

This pulse is referred to as the *Dirac Delta Pulse* δ(t) and its transform is given by:

$$\mathrm{F}\{\delta(t)\} \equiv \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1$$

This means that this single impulse contains all frequencies.

From this result we may conclude that the transfer function of a linear system may be specified by either its frequency response in the frequency domain.




The same result can be achieved by taking the Fourier transform of the impulse response.

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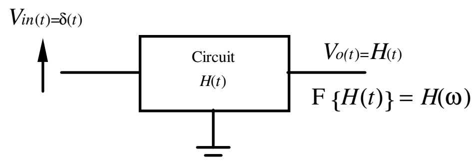
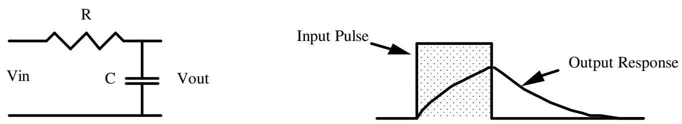
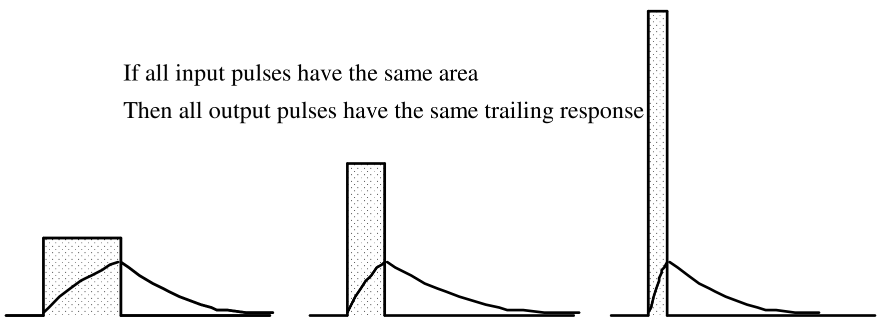



## Impulse Response of a Low Pass Filter

Since the Fourier transform of a Dirac delta pulse contains all frequencies, it should come as no surprise that the system response to a unit impulse completely describes the dynamic behavior of a linear system.




An interesting phenomenon occurs when a system, such as an RC network, is excited by pulse with constant area but varying amplitude:




As the input pulse is changed, the leading edge of the output response changes, while the trailing edge does not. If a delta pulse stimulates the circuit, there is only the trailing response, known the impulse response. The Fourier transform of the impulse response *h(t)*, is the frequency response *H(f)*.

From the Appendix, we note that the impulse response for a simple *RC* network is: $h(t) = \frac{1}{RC} e^{-t/RC}$ for $t \geq 0$.

## Fourier Transform of the Impulse Response

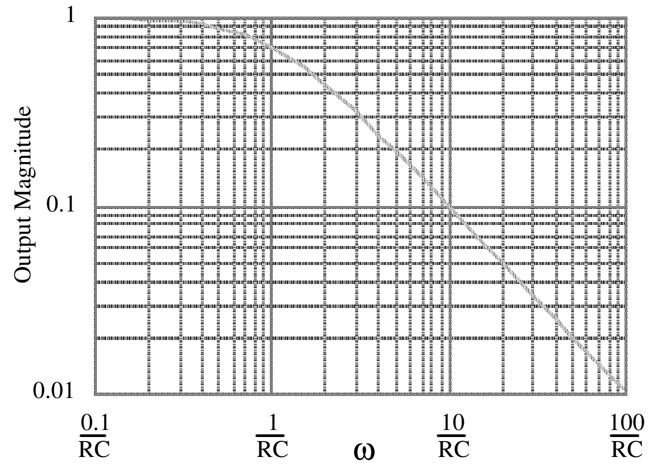
Taking the Fourier transform of the impulse response, we arrive at the frequency response:

$$\mathrm{F}\{h(t)\} = H(\omega) = \int_0^\infty \frac{1}{RC} e^{-t/RC} e^{-j\omega t} dt$$

$$= \frac{1}{RC} \left( \frac{-1}{\frac{1}{RC} + j\omega} \right) e^{-(1/RC + j\omega)t} \Big|_0^\infty$$

$$= \frac{1}{1 + j\omega RC}$$

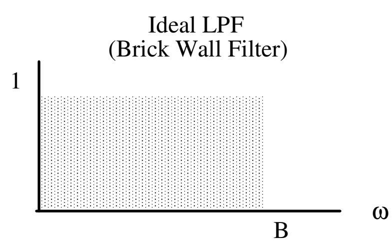
Notice that the impulse response is a complex function. This means this filter response has both a magnitude and phase components. The amplitude response resembles:




It is quite easy to determine from the filter response, that the roll off rate is 20 dB per decade. For many applications, this rate is not high enough, so multiple filter stages, or poles are often connected in series. The ideal or perfect filter would have an infinite roll off rate. It is not possible to construct such a filter. However, like all ideals, it defines a limiting condition or boundary and is therefor quite useful in analyzing and predicting circuit behavior.

## Ideal Low Pass Filter

An ideal low pass filter has unity gain and bandwidth $B$. The cutoff frequency is given by: $B = 2\pi f_c$.




$$H(\omega) = 1 \qquad 0 \le \omega \le B$$

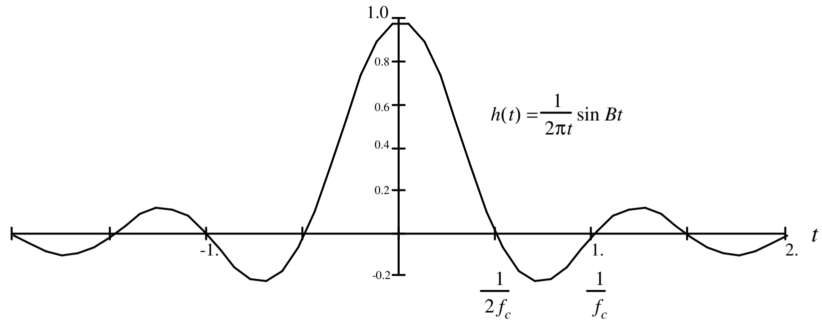
The impulse response of this filter is known as a sinc function. Since the function has even symmetry, only the real part of the transform is valid:

$$h(t) = \mathrm{F}^{-1}\{H(\omega)\} = \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^B H(\omega)\{\cos(\omega t) + j\sin(\omega t)\}d\omega\right\}$$

$$= \frac{1}{2\pi}\left[\frac{1}{t}\sin \omega t\right]_0^B$$

$$= \frac{1}{2\pi t}\sin Bt$$

A plot of this function resembles:




A more rigorous approach would be to consider the frequency domain as also containing negative frequencies, in which case the result would be:

$$h(t) = \mathrm{F}^{-1}\{H(\omega)\} = \frac{1}{2\pi}\int_{-B}^{B} H(\omega)\{\cos(\omega t) + j\sin(\omega t)\}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{t}\sin\omega t - j\frac{1}{t}\cos\omega t\right]_{-B}^{B}$$

$$= \frac{1}{\pi t}\sin Bt$$

The result is nearly identical to that already obtained except that the magnitude is twice as large.

## The Inverse Fourier Transform:

The inverse transform is used to convert from the frequency domain back to the time domain.

$$\mathrm{F}^{-1}\{F(\omega)\} \equiv f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

The additional term 1/2π in the inverse transform, comes from the convention that the period $t$ is related to frequency:

$$t = \frac{1}{f} = \frac{2\pi}{\omega}$$

The transform can be written in many ways. One variation takes advantage of Euler's Identity, which is used to transform complex exponents to complex trigonometry:

$$e^{j\varphi} = \cos(\varphi) + j\sin(\varphi)$$

Therefor an alternate form of the Fourier transform is:

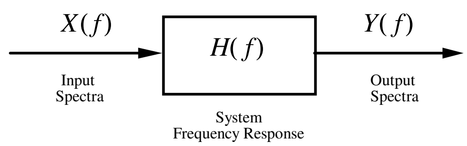
$$\mathrm{F}\{f(t)\} \equiv F(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

From this we observe that the Fourier transform of a real time domain function has a complex frequency domain representation.

## Convolution

The input and output spectra of any linear system is related by the following product:

$$Y(\omega) = X(\omega) \times H(\omega) \quad \text{or} \quad Y(f) = X(f) \times H(f)$$



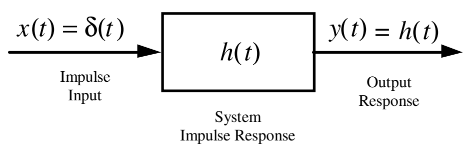

For example, if a sinewave sin(ωt) was applied to a simple *RC* network, we would expect the output $Y(\omega)$ to be:

$$Y(\omega) = H(\omega) \times X(\omega) = \frac{1}{1 + j\omega RC} \sin(\omega t)$$

The equivalent relationship in the time domain is known as a convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(z) h(t - z) dz$$

In the special case where the input is a delta pulse, the output is by definition the impulse response:




Actual filters operate in the time domain. In many cases, particularly with analog circuits, it is quite difficult to perform the time domain convolutions in order to predict the output response. However, it is a lot easier to perform convolutions in linear discrete systems such as found in digital filters. These can be characterized by the unit sample response:

$$y(n) = x(n) * h(n)$$

where the discrete convolution is given by:

$$y_i = \sum_{k=0}^{i} x_k h_{i-k}$$

This latter expression opens the world of DSP and makes it possible to implement a wide range of filter types in the digital domain.

# Assignment Questions

## Quick Quiz

1. The [leading, trailing] output response of an *RC* network remains constant as long as the input pulse area remains constant.

## Analytical Problems

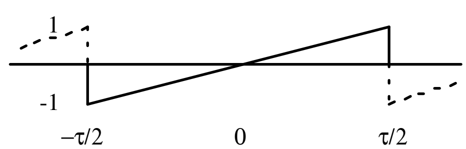
1. Find the Fourier transform of a flat-topped pulse by invoking symmetry.

2. Prove that

$$b_n = \frac{1}{\pi}\int \quad (x)\sin(nx)\,dx$$

in the harmonic Fourier Transform.

3. Find the continuous and discrete Fourier Transforms for the following ramp signal:



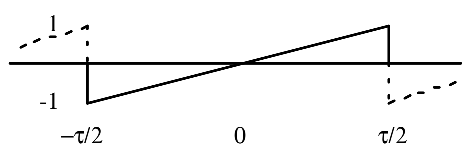

4. Plot the phase response for a single pole lowpass filter over the frequency interval 0.1/RC to 100/RC.

## Composition Questions

1. What is the difference in the frequency spectrum of a single pulse and a series of pulses?

### Solution to Analytical Problem 3

5. Find the continuous and discrete Fourier Transforms for the following ramp signal:




$$f(t) = \frac{2}{\tau}t$$

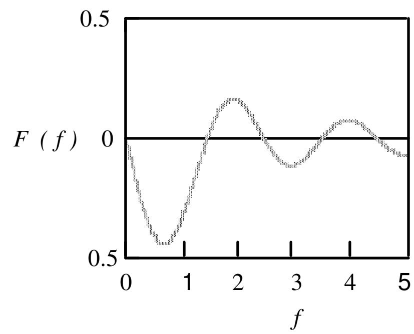
Notes

$$\mathrm{F}\{f(t)\} = F(\omega) = \int_{-\tau/2}^{\tau/2} \frac{2}{\tau} t e^{-j\omega t} dt$$

$$= \frac{1}{\tau}\left[\frac{e^{-j\omega t}}{(-j\omega t)^2}(-j\omega t - 1)\right]_{-\tau/2}^{\tau/2}$$

$$= j\frac{4}{(\tau\omega)^2}\left[\frac{-2}{\tau}\sin\left(\frac{\omega\tau}{2}\right) + \omega\cos\left(\frac{\omega\tau}{2}\right)\right]$$

Plotting this as a function of $f$ instead of ω, we obtain:



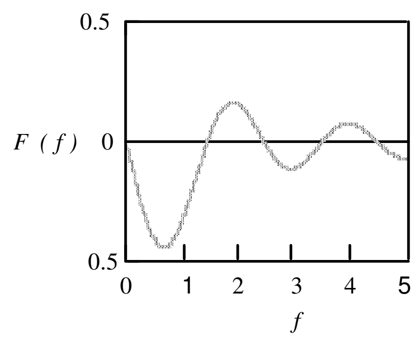

The solution can also be achieved by invoking symmetry:

$$\mathrm{F}\{f(t)\} = F(\omega) = -j\int_{-\tau/2}^{\tau/2} \frac{2}{\tau} t \sin(\omega t) dt$$

$$= -j\frac{2}{\tau}\left[\frac{1}{\omega^2}\sin(\omega t) - \frac{t}{\omega}\cos(\omega t)\right]_{-\tau/2}^{\tau/2}$$

$$= j\frac{2}{\tau\omega}\left[\tau\cos\left(\frac{\omega\tau}{2}\right) - \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)\right]$$

Although this looks quite different from the previous solution, it is in fact identical.




The discrete solution is:

By inspection: $a_0 = 0 \quad a_n = 0$

$$b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} \frac{2}{\tau} t \sin\left(\frac{2\pi nt}{\tau}\right) dt$$

$$= \left(\frac{2}{\tau}\right)^2 \left[\left(\frac{\tau}{2\pi n}\right)^2 \sin\left(\frac{2\pi nt}{\tau}\right) - \left(\frac{\tau}{2\pi n}\right) t \cos\left(\frac{2\pi nt}{\tau}\right)\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{2}{\pi n}(-1)^{n+1}$$

Therefore, putting it all together:

$$F(\omega) = \sum_{n=1}^{\infty} \frac{2}{\pi n}(-1)^{n+1} \sin\left(\frac{2\pi nt}{\tau}\right)$$

Or expanding it as a series:

$$F(\omega) = \frac{2}{\pi}\sin(\omega t) - \frac{1}{\pi}\sin(2\omega t) + \frac{2}{3\pi}\sin(3\omega t) - \frac{1}{2\pi}\sin(4\omega t) + \frac{2}{5\pi}\sin(5\omega t) - \cdots$$

This again looks different from the previous solutions, but the results are identical for the discrete frequency values.

_____ Notes _____

# For Further Research

http://www.spd.eee.strath.ac.uk/~interact/fourier/

http://husky.northern-hs.ga.k12.md.us/4ier.html

http://capella.dur.ac.uk/doug/fourier.html

http://www.relisoft.com/freq.html

http://www.imaging.org/tutorial/jpgdct1.html

http://www.cage.curtin.edu.au/mechanical/info/vibrations/tutor.htm