Appendix 4, Laplace Transforms

Circuit Review

All (or almost all) electrical components exhibit the quality called impedance. Impedance is an electrical interaction between electrons flowing through a component, and the resulting electromagnetic fields. It is also the ratio of voltage over current, these three attributes being related by Ohms Law.

The power dissipated in any component is defined as:

$$P = IE\cos\theta$$

Where the angle θ is the phase angle difference between the voltage and current. The simplest form of impedance is resistance. Its impedance is generally constant and independent of frequency. Since the voltage and current are in phase, power is dissipated.

$$e = iR \quad \text{and} \quad X_R = R$$

A much more complex component is the inductor. In this case, there is a modification of the voltage-current relationship. It turns out that if sine waves are used, the phase relationship is orthogonal, resulting in no power dissipation.

$$e = L\frac{di}{dt}$$

$$\text{if} \quad i = \sin\omega t$$

$$\text{then} \quad e = L\frac{d}{dt}\sin\omega t = L\omega\cos\omega t$$

Furthermore, the magnitude of the impedance is directly proportional to frequency.

$$X_L = 2\pi fLj$$

The term j is used to create an axis orthogonal to X_R to help characterize the current-voltage relationship. Hence a distinction is made between 'real power' and 'reactive power'.

The capacitor is a very complex device, even more so than an inductor, since there is a displacement current but no actual electrical current flow through the component. Again there is an orthogonal relationship between voltage and current, but it is exactly opposite to that of the inductor.

$$e = \frac{1}{C}\int \quad dt$$

$$\text{if} \quad i = \sin \omega t$$

$$\text{then} \quad e = \frac{1}{C}\int \sin \omega t \; dt = \frac{-1}{\omega C}\cos \omega t$$

The impedance magnitude of the capacitor is inversely proportional to frequency.

$$X_C = \frac{1}{2\pi f C j}$$

The *j* term in the denominator accounts for the phase reversal since $1/j = -j$. To simplify the notation, it is often convenient to let $2\pi f j = \omega j = s$.

Consequently:

$$X_L = Ls \qquad \text{and} \qquad X_C = \frac{1}{Cs}$$

In actuality, $s = \sigma + j\omega$, but ideal capacitors and inductors have no real component and so $\sigma = 0$.

It is often advantageous to convert from the time domain to the frequency domain or vice versa. It would also be a great convenience to apply a transformation to the time domain expressions so as to eliminate any differential or integral operator, and reduce any analysis to 'simple' algebra.

One such transformation, which meets both of these objectives, is the Laplace Transform.

Laplace Transform

One of the major applications of the Laplace Transform in engineering is in determining the transient response of networks. Fourier Transforms are less effective in this because:

- for some functions, such as the unit step function, the Fourier integral does not converge
- the response function appears as an integral that may be difficult to evaluate
- the circuit must be initially relaxed when performing Fourier analysis

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

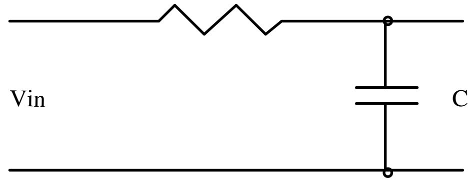
Taking the inverse transform is somewhat more difficult, and may be found by performing the following integral or by consulting transform tables:

$$\mathcal{L}^{-1}\{F(s)\} \equiv f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, dt$$

$$\text{where} \qquad s = \sigma + j\omega$$

Example

Let's see how we can use these mathematical tools to analyze circuit behavior. Given the following circuit, analyze its behavior:



We may write the output voltage as:

$$V_{out} = \frac{1}{C}\int i\,dt$$

$$\text{since} \qquad i = \frac{V_{in} - V_{out}}{R}$$

$$\text{then} \qquad V_{out} = \frac{1}{C}\int \frac{V_{in} - V_{out}}{R}\,dt$$

$$V_{out} = \frac{1}{RC}\int \left(V_{in} - V_{out}\right)dt$$

Differentiating both sides, and rewriting the equation we obtain:

$$\frac{d\,V_{out}}{dt} + \frac{1}{RC}V_{out} = \frac{1}{RC}V_{in}$$

This is a nonhomogeneous first order linear differential equation. Applying the general solution for such equations, we obtain:

$$V_{out} = e^{-t/RC}\int \frac{1}{RC}V_{in}\,e^{t/RC}\,dt + K\,e^{-t/RC}$$

Where K is evaluated at the initial condition. Let's find the circuit characteristic equation, by letting $Vin = \delta(t)$.

Impulse Response

$$V_{out} = e^{-t/RC}\underbrace{\int \frac{1}{RC}\delta(t)e^{t/RC}\,dt}_{=\ 0\text{ when t } \neq\ 0} + K\,e^{-t/RC}$$

$$V_{out} = e^{-t/RC}\left[\frac{1}{RC}e^{t/RC}\right]^{t=0} + K\,e^{-t/RC}$$

$$= \frac{1}{RC}e^{-t/RC} + K\,e^{-t/RC}$$

In order to find the value of *K*, we have to assess the function at the initial condition.

$$V_{out} = 0 \qquad \text{when} \qquad t < 0$$

$$0 = 0 + Ke^{<0/RC}$$

$$\therefore K = 0$$

$$V_{out} = \frac{1}{RC} e^{-t/RC}$$

Let's now evaluate the circuit response to a unit step function at $t = 0$.

Step Response

The step function is generally designated by *u(t)*. Letting *Vin = u(t)*:

$$\begin{aligned} V_{in} &= 0 \qquad \text{for} \qquad t < 0 \\ &= 1 \qquad \text{for} \qquad t \geq 0 \end{aligned}$$

$$\begin{aligned} V_{out} &= 0 \qquad \text{for} \qquad t \leq 0 \\ &= ? \qquad \text{for} \qquad t > 0 \end{aligned}$$

Applying these conditions we obtain:

$$\begin{aligned} V_{out} &= e^{-t/RC} \int \frac{1}{RC} u(t) e^{t/RC} dt + K e^{-t/RC} \\ &= e^{-t/RC} \frac{1}{RC} \frac{e^{t/RC}}{\frac{1}{RC}} + K e^{-t/RC} \\ &= 1 + K e^{-t/RC} \end{aligned}$$

To find the value of *K*, we simply apply the initial conditions:

$$0 = 1 + Ke^{0/RC}$$

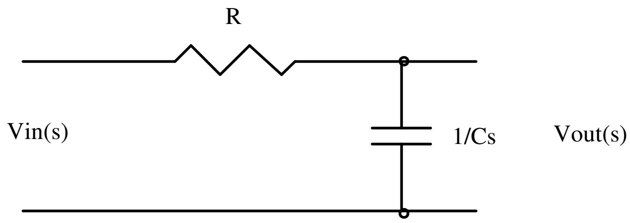
$$\therefore K = -1$$

Therefor the final solution is:

$$V_{out} = 1 - e^{-t/RC}$$

There has to be an easier was of doing this! Let's try Laplace Transforms.

In the s domain, the circuit resembles:



The output is simply a divider:

$$V_{out} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} V_{in} = \frac{1}{RCs + 1} V_{in}$$

$$= \frac{1}{RC} \frac{1}{s + \frac{1}{RC}} V_{in}$$

Impulse Response

The Laplace Transform of the impulse function is:

$$\mathcal{L}\{\delta(t)\} \equiv 1$$

Therefor the output is:

$$V_{out} = \frac{1}{RC}\left(\frac{1}{s + \frac{1}{RC}}\right)(1)$$

Now we could take the inverse transform ourselves:

$$V_{out}(t) = \frac{1}{2\pi j} \frac{1}{RC} \int_{\sigma - j\infty}^{\sigma + j\infty} \left(\frac{1}{s + \frac{1}{RC}}\right)(1) e^{st}\, dt$$

Or simply look up the solution from the inverse transform tables:

$$f[\delta(t)] = \frac{1}{RC} e^{-t/RC}$$

STEP RESPONSE

The Laplace Transform value of a step function is:

$$\mathcal{L}\{u(t)\} \equiv \frac{1}{s}$$

So now the output can be written:

$$V_{out} = \frac{1}{RC}\left(\frac{1}{s+\frac{1}{RC}}\right)\left(\frac{1}{s}\right)$$

And from the tables, we obtain:

$$V_{out} = \left(\frac{1}{RC}\right)\left(\frac{1}{\frac{-1}{RC}-0}\right)\left(e^{-t/RC} - e^{0}\right)$$

$$= 1 - e^{-t/RC}$$

We should not be overly concerned with the need to consult tables, since we must do this for solutions to integrals as well. Of the two methods, this method is generally the easier.