CONTENTS

15.0 Transmission Line Theory

Data Transmission Lines and Their Characteristics by National Semiconductor

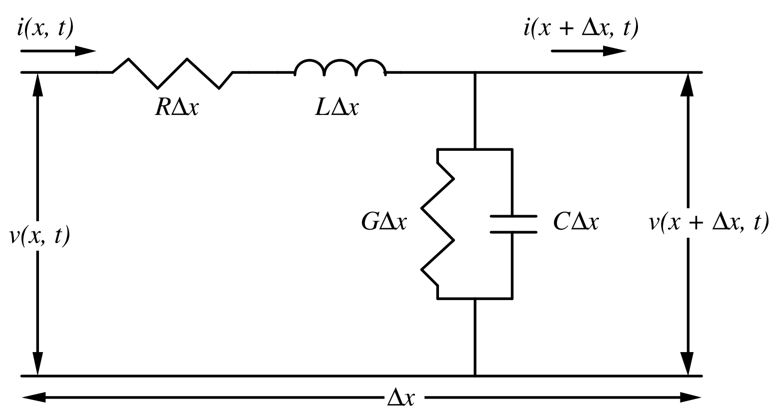
It may seem strange to have a section on transmission lines in a wireless document. However, all radio wireless systems require some sort of antenna and transmission system.

The following analysis requires two assumptions:

- a transmission line can be decomposed into small, distributed passive electrical elements
- these elements are independent of frequency

These two assumptions limit the following analysis to frequencies up to the low MHz region. The second assumption is particularly difficult to defend since it is well known that the resistance of a wire increases with frequency because the conduction cross-section decreases. This phenomenon is known as the skin effect and is not easy to evaluate.

EQUIVALENT CIRCUIT



Distributed Element Model of a Transmission Line

The purpose behind the following mathematical manipulation is to obtain an expression that defines the voltage (or current) at any time along any portion of the transmission line. Later, this analysis will be extended to include the frequency domain.

Recall the characteristic equations for inductors and capacitors:

$$v = L\frac{\partial i}{\partial t} \quad \text{and} \quad i = C\frac{\partial v}{\partial t}$$

Applying KVL in the above circuit, we obtain:

$$v(x,t) = R\Delta x\, i(x,t) + L\Delta x \frac{\partial i}{\partial t}(x,t) + v(x+\Delta x,t)$$

Rearranging:

$$v(x,t) - v(x+\Delta x,t) = R\Delta x\, i(x,t) + L\Delta x \frac{\partial i}{\partial t}(x,t)$$

But the LHS† of the above equation, represents the voltage drop across the cable element Δv, therefor:

$$\Delta v = R\Delta x\, i(x,t) + L\Delta x \frac{\partial i}{\partial t}(x,t)$$

Dividing through by Δx, we obtain:

$$\frac{\Delta v}{\Delta x} = R\, i(x,t) + L \frac{\partial i}{\partial t}(x,t)$$

The LHS is easily recognized as a derivative. Simplifying the notation:

$$\frac{\partial v}{\partial x} = R\, i + L \frac{\partial i}{\partial t}$$

This expression has both current and voltage in it. It would be convenient to write the equation in terms of current or voltage as a function of distance or time.

The first step towards meeting this goal is to take the derivative with respect to x:

$$\frac{\partial^2 v}{\partial x^2} = \overbrace{R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial x \partial t}}^{\text{eliminate } i \text{ and } x} \qquad 15-1$$

The next step is to eliminate the current terms, leaving an expression with voltage only. The change in current along the line is equal to the current being shunted across the line through C and G. By applying KCL in the circuit, we obtain the necessary information:

$$\frac{\partial i}{\partial x} = Gv + C \frac{\partial v}{\partial t} \qquad 15-2$$

† Left Hand Side

_____ Notes _____

Taking the derivative with respect to time, we obtain:

$$\frac{\partial^2 i}{\partial x \partial t} = G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2} \qquad 15-3$$

Substituting equations 13 – 2 and 13 – 3 into 13 – 1, we obtain the desired simplification:

$$\frac{\partial^2 v}{\partial x^2} = R\left[Gv + C\frac{\partial v}{\partial t}\right] + L\left[G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2}\right]$$

Collecting the terms, we obtain:

$$\frac{\partial^2 v}{\partial x^2} = RGv + (RC + LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2}$$

This equation is known as the transmission line equation. Note that it has voltage at any particular location as a function of time.

Similarly for current, we obtain:

$$\frac{\partial^2 i}{\partial x^2} = RGi + (RC + LG)\frac{\partial i}{\partial t} + LC\frac{\partial^2 i}{\partial t^2}$$

Solving the Transmission Line Equation

A mathematician would solve the transmission line equation for v. Historically, this is done by assuming a solution for v, substituting it into the equation, and observing whether the result made any sense.

An engineer would follow a similar procedure by making an "educated guess" based on some laboratory experiments, as to what the solution might be. Today there are more sophisticated techniques used to find solutions. In this respect, the engineer may lag behind the mathematician by several centuries in finding applications for mathematical tools.

For the transmission line equation, we shall guess that the solution for the voltage function is of the form:

$$v(t) = e^{j\omega t} e^{-\gamma x} \qquad 15-4$$

The first term simply represents a unity vector rotating at an angular velocity of ω radians per second, in other words, a sinewave. The second term denotes the sinusoid as modified by the transmission line. Its amplitude decaying exponentially with distance.

The sine wave is used as a signal source because it is easy to generate, and manipulate mathematically. Euler's Identity shows the relationship between exponential notation and trigonometric functions:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

If we let γ be a complex quantity, we can also include any phase changes which occur as the signal travels down the line.

let $$\gamma = \alpha + j\beta$$

Then $$v(t) = e^{j\omega t} e^{-(\alpha + j\beta)x} = \underbrace{e^{-\alpha x}}_{\text{amplitude}} \underbrace{e^{(\omega t - \beta x)j}}_{\text{frequency and phase}}$$

The amplitude term denotes the exponential decay of the signal. α is known as the attenuation coefficient and is expressed in Nepers per meter. The second term denotes the frequency and phase of the signal. The β component is known as the phase shift coefficient, and is expressed in radians per meter.

Substituting our educated guess into the transmission line equation, we obtain:

$$\overbrace{\frac{\partial^2}{\partial x^2}\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}}^{LHS} = \overbrace{\underbrace{RG\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}}_{\text{1st Term}} + \underbrace{(RC + LG)\frac{\partial}{\partial t}\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}}_{\text{2nd Term}} + \underbrace{LC\frac{\partial^2}{\partial t^2}\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}}_{\text{3rd Term}}}^{RHS}$$

The LHS of the above equation can be simplified as follows:

$$\frac{\partial^2}{\partial x^2}\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\} = \frac{\partial}{\partial x}\left\{(\alpha + j\beta)e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}$$
$$= (\alpha + j\beta)^2 e^{j\omega t} e^{-(\alpha + j\beta)x}$$

The 1st term of the RHS does not require any simplification:

$$\text{1st Term } = RG\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}$$

The 2nd term of the RHS can be simplified:

$$\text{2nd Term } = (RC + LG)\frac{\partial}{\partial t}\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\} = (RC + LG)j\omega\left\{e^{j\omega t} e^{-(\alpha + j\beta)x}\right\}$$

The 3rd term of the RHS can also be simplified:

$$\begin{aligned}\text{3rd Term } &= LC\frac{\partial^2}{\partial t^2}\left\{e^{j\omega t}e^{-(\alpha+j\beta)x}\right\}\\ &= LC\frac{\partial}{\partial t}\left\{j\omega e^{j\omega t}e^{-(\alpha+j\beta)x}\right\}\\ &= -LC\omega^2 e^{j\omega t}e^{-(\alpha+j\beta)x}\end{aligned}$$

Rewriting the equation in terms of the simplifications:

$$\overbrace{(\alpha+j\beta)^2 e^{j\omega t}e^{-(\alpha+j\beta)x}}^{LHS} = \overbrace{\underbrace{RG\left\{e^{j\omega t}e^{-(\alpha+j\beta)x}\right\}}_{\text{1st Term}} + \underbrace{(RC+LG)j\omega\left\{e^{j\omega t}e^{-(\alpha+j\beta)x}\right\}}_{\text{2nd Term}} \underbrace{-LC\omega^2 e^{j\omega t}e^{-(\alpha+j\beta)x}}_{\text{3rd Term}}}^{RHS}$$

Note that each of the terms contains the original exponential:

$$v(t) = e^{j\omega t}e^{-(\alpha+j\beta)x} = e^{-\alpha x}e^{(\omega t-\beta x)j}$$

Eliminating the common term, we obtain:

$$\begin{aligned}(\alpha+j\beta)^2 &= RG + (RC+LG)j\omega - LC\omega^2\\ &= (R+j\omega L)(G+j\omega C)\end{aligned}$$

or

$$\alpha + j\beta = \gamma = \sqrt{(R+j\omega L)(G+j\omega C)} \qquad 15-5$$

This result tells us that our assumed solution describing how voltage travels down a transmission line:

$$v(t) = e^{j\omega t}e^{-\gamma x}$$

is correct if equation 13 - 5 is true. Since α and β are arbitrary constants and can take any form, it folows that equations 13 - 4 and 13 - 5 are valid.

Solving for α and β, can be somewhat tedious unless one has a calculator capable of manipulating complex numbers. By displaying the result in polar form, the magnitude and phase can be readily observed. The magnitude of γ can also be found by squaring all of the components in the above expression, and then taking the root. This is an application of the Pythagorean theorem.

$$|\gamma| = \sqrt{\alpha^2+\beta^2} = \sqrt{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}}$$

The angle is found by taking one half of the sum of the inverse tangents of the individual vector components:

$$\varphi = \frac{1}{2}\left[\tan^{-1}\left(\frac{\omega L}{R}\right) + \tan^{-1}\left(\frac{\omega C}{G}\right)\right]$$

Consequently, we can now write α and β in terms of the distributed component values:

$$\alpha = |\gamma|\cos(\varphi) \text{ nepers/meter}$$
$$\beta = |\gamma|\sin(\varphi) \text{ radians/meter}$$

EXAMPLE

If we examine a hypothetical transmission line, it may become more readily apparent as to what these mathematical expressions mean. Given a cable with the following [unrealistic] characteristics:

R = .568 Ω/meter

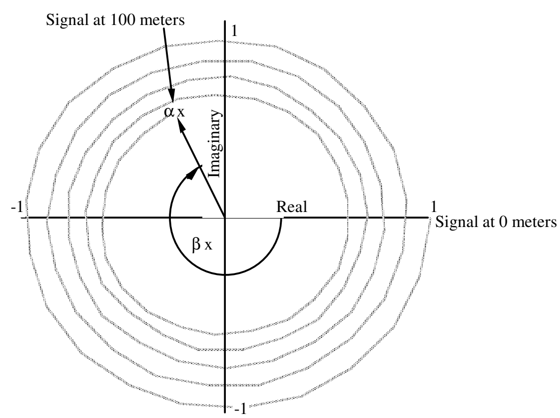
L = 234 x 10^{-9} H/m

C = 93.5 x 10^{-12} F/m

G = 1 x 10^{-9} mhos/m

How is a 10 MHz signal affected as it moves down a 100-meter cable? The following illustration is a plot of:

$$v(t) = e^{j\omega t}e^{-\gamma x} \quad \text{where} \quad \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$



Magnitude and phase as a 10 MHz signal passes down a certain transmission line

Lossless Transmission Line

Signal loss occurs by two basic mechanisms: signal power can be dissipated in a resistor [or conductance] or signal currents may be shunted to an AC ground via a reactance. In transmission line theory, a lossless transmission line does not dissipate power. Signals, will still gradually diminish however, as shunt reactances return the current to the source via the ground path.

For the power loss to equal zero, $R = G = 0$. This condition occurs when the transmission line is very short. An oscilloscope probe is an example of a very short transmission line. The transmission line equation reduces to:

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \qquad 15 – 6$$

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}$$

To determine how sinusoidal signals are affected by this type of line, we simply substitute a sinusoidal voltage or current into the above expressions and solve as before, or we could take a much simpler approach. We could start with the solution for the general case:

$$\alpha + j\beta = \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Let $R = G = 0$, and simplify:

$$\begin{aligned}\alpha + j\beta &= \sqrt{(j\omega L)(j\omega C)} \\ &= \omega(LC)^2 j\end{aligned}$$

Equating the real and imaginary parts:

$$\begin{aligned}\alpha &= 0 \\ \beta &= \omega\sqrt{LC}\end{aligned} \qquad 15 – 7$$

This expression tells us that a signal travelling down a lossless transmission line, experiences a phase shift directly proportional to its frequency.

PHASE VELOCITY

A new parameter, known as phase velocity, can be extracted from these variables:

$$V_p = \frac{1}{\sqrt{LC}} = \frac{\omega}{\beta} \text{ meters/second} \qquad 15 – 8$$

Phase velocity is the speed at which a fixed point on a wavefront, appears to move. In the case of wire transmission lines, it is also the velocity of propagation., typically: $0.24c < V_p < 0.9c$.

The distance between two identical points on a wavefront is its wavelength (λ) and since one cycle is defined as 2π radians:

$$\lambda = \frac{2\pi}{\beta} \quad \text{and} \quad \omega = 2\pi f$$

therefore: $V_p = \lambda f$

In free space, the phase velocity is 3 x 10^8 meters/sec, the speed of light. In a cable, the phase velocity is somewhat lower because the signal is carried by electrons. In a waveguide transmission line, the phase velocity exceeds the speed of light.

Substituting 13 – 8 into 13 – 5, we obtain:

$$v = v_o \cos\left(\omega\left[t - \sqrt{LC}\, x\right]\right)$$

Or 15 – 9

Distortionless Transmission Line

A distortionless transmission line is not the same as a lossless one. A lossless line modifies the phase characteristics of the signal but does not consume power. A distortionless line does not distort the signal phase, but does introduce a signal loss. Since common transmission lines are not super conductors, a distortionless line does produce attenuation distortion.

Phase distortion does not occur if the phase velocity V_p is constant at all frequencies.

By definition, a phase shift of 2π radians occurs over one wavelength λ.

Since: $$V_p = \lambda f \quad \lambda = \frac{2\pi}{\beta} \quad \text{and} \quad f = \frac{\omega}{2\pi}$$

Then: $$V_p = \frac{2\pi}{\beta} \times \frac{\omega}{2\pi} = \frac{\omega}{\beta}$$

This tells us that in order for phase velocity V_p to be constant, the phase shift coefficient β, must vary directly with frequency ω.

Recall: $$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

The problem now is to find β. This can be done as follows:

$$\gamma = \sqrt{\left(\frac{R + j\omega L}{j\omega L}\right)(j\omega L)\left(\frac{G + j\omega C}{j\omega C}\right)(j\omega C)}$$
$$= j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}}\sqrt{1+\frac{G}{j\omega C}}$$

The 2nd and 3rd roots can be expanded by means of the Binomial Expansion. Recall:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \quad ...$$

In this instance n = 1/2. Since the contribution of successive terms diminishes rapidly, γ is expanded to only 3 terms:

$$\gamma \approx j\omega\sqrt{LC}\left\{\left[1+\frac{1}{2}\frac{R}{j\omega L}-\frac{1}{8}\left(\frac{R}{j\omega L}\right)^2\right]\left[1+\frac{1}{2}\frac{G}{j\omega C}-\frac{1}{8}\left(\frac{G}{j\omega C}\right)^2\right]\right\}$$

$$\approx j\omega\sqrt{LC}\left\{\begin{array}{l} 1+\frac{1}{2}\frac{G}{j\omega C}-\frac{1}{8}\left(\frac{G}{j\omega C}\right)^2+\frac{1}{2}\frac{R}{j\omega L}-\frac{1}{4}\frac{RG}{LC\omega^2} \\ -\frac{1}{16}\left(\frac{R}{j\omega L}\right)\left(\frac{G}{j\omega C}\right)^2-\frac{1}{8}\left(\frac{R}{j\omega L}\right)^2 \\ -\frac{1}{16}\left(\frac{R}{j\omega L}\right)^2\left(\frac{G}{j\omega C}\right)+\frac{1}{64}\left(\frac{R}{j\omega L}\right)^2\left(\frac{G}{j\omega C}\right)^2 \end{array}\right\}$$

Since $\gamma = \alpha + j\beta$, equate the imaginary terms to find β.

$$\beta \approx \omega\sqrt{LC}\left\{1+\underbrace{\frac{1}{8}\left(\frac{G}{\omega C}\right)^2-\frac{1}{4}\frac{RG}{LC\omega^2}+\frac{1}{8}\left(\frac{R}{\omega L}\right)^2}_{\text{Difference of Squares}}+\underbrace{\frac{1}{64}\left(\frac{R}{\omega L}\right)^2\left(\frac{G}{\omega C}\right)^2}_{\text{Very Small}}\right\}$$
$$\approx \omega\sqrt{LC}\left\{1+\frac{1}{8}\left(\frac{R}{\omega L}-\frac{G}{\omega C}\right)^2\right\}$$

Note that if $\frac{R}{\omega L} = \frac{G}{\omega C}$ then: $\beta \approx \omega\sqrt{LC}$

From this we observe that β is directly proportional to ω. This means that the requirement for distortionless transmission is:

$$RC = LG$$

If we equate the real terms, we obtain:

$$\alpha \approx \sqrt{RG}$$

The Frequency Domain

Signal analysis is often performed the frequency domain. This tells us how the transmission line affects the spectral content of the signals they are carrying.

To determine this, it is necessary to find the Fourier Transform of the transmission line equation.

Recall: $$\frac{\partial^2 v}{\partial x^2} = RGv + (RC + LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2}$$

and recall the Fourier Transform:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt$$

To prevent this analysis from 'blowing up', we must put a stipulation on the voltage function namely, that it vanishes to zero at an infinite distance down the line. This comprises a basic boundary condition.

$$\text{let} \quad v \to 0 \quad \text{as} \quad x \to \infty$$

This stipulation is in agreement with actual laboratory experiments. It is well known that the signal magnitude diminishes as the path lengthens.

Likewise, a time boundary condition, that the signal was zero at some time in the distant past and will be zero at some time in the distant future, must be imposed.

$$\text{let} \quad v \to 0 \quad \text{as} \quad t \to \infty$$

Although engineers have no difficulty imposing these restrictions, mathematical purists, are somewhat offended. For this and other reasons, other less restrictive transforms have been developed. The most notable in this context, is the Laplace transform, which does not have the same boundary conditions.

Having made the necessary concessions in order to continue our analysis, we must find the Fourier Transform corresponding to the following terms:

$$\mathcal{F}\{v\}, \quad \mathcal{F}\left\{\frac{\partial v}{\partial t}\right\} \quad \text{and} \quad \mathcal{F}\left\{\frac{\partial^2 v}{\partial t^2}\right\}$$

Let: $$\mathcal{F}\{v\} = V$$

Then applying the transform on the derivative, we obtain:

$$\mathcal{F}\left\{\frac{\partial v}{\partial t}\right\} = \int_{-\infty}^{\infty} e^{-j\omega t}\frac{\partial v}{\partial t}\,dt$$

This equation can be solved by using integration by parts:

$$\int u\,dv = uv - \int v\,du$$

let $u = e^{-j\omega t}$ $\quad \therefore du = -j\omega e^{-j\omega t}\,dt$

and $dv = \frac{\partial v}{\partial t}$ $\quad \therefore v = v$

$$\therefore \mathcal{F}\left\{\frac{\partial v}{\partial t}\right\} = e^{-j\omega t} v \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\left(-j\omega e^{-j\omega t}\right)dt$$

Applying the boundary conditions when t goes to infinity makes the 1st term disappear.

$$\therefore \mathcal{F}\left\{\frac{\partial v}{\partial t}\right\} = j\omega \int_{-\infty}^{\infty} e^{-jwt}\,v\,dt$$

Note that the resulting integral is simply the Fourier Transform. In other words:

$$\mathcal{F}\left\{\frac{\partial v}{\partial t}\right\} = j\omega\,\mathcal{F}\{v\} = j\omega V$$

Similarly:
$$\mathcal{F}\left\{\frac{\partial^2 v}{\partial t^2}\right\} = (j\omega)^2\,\mathcal{F}\{v\} = (j\omega)^2 V$$

We can now write the transmission line equation in the frequency domain:

$$\frac{\partial^2 V}{\partial x^2} = RGV + (RC + LG)j\omega\,V + LC(j\omega)^2\,V$$

Where: $V = V(\omega) = \mathcal{F}\{v(t)\}$

Rearranging the terms, we obtain:

$$\frac{\partial^2 V}{\partial x^2} = \left[RG + (RC + LG)j\omega + (j\omega L)(j\omega C)\right]V$$

Or
$$\frac{\partial^2 V}{\partial x^2} = \left[(R + j\omega L)(G + j\omega C)\right]V$$

Since
$$\sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta = \gamma$$

Then $$\frac{\partial^2 V}{\partial x^2} = \gamma^2 V$$

Or $$\frac{\partial^2 V}{\partial x^2} - \gamma^2 V = 0$$

This represents the most general form of the transmission line equation in the frequency domain. This equation must now be solved for V to observe how voltage (or current) varies with distance and frequency. This can be done by assuming a solution of the form:

$$V = \underbrace{Ae^{-\gamma x}}_{\substack{\text{forward} \\ \text{wave}}} + \underbrace{Be^{\gamma x}}_{\substack{\text{reverse} \\ \text{wave}}}$$

These terms represent an exponential decay as the signal travels down the transmission line. If we ignore any reflections, assuming that the cable is infinitely long or properly terminated, this simplifies to:

$$V = V_o e^{-\gamma x}$$

To verify whether this assumption is correct, substitute it into the equation, and see if a contradiction occurs. If there is no contradiction, then our assumption constitutes a valid solution.

$$\frac{\partial^2}{\partial x^2} V_o e^{-\gamma x} - \gamma^2 V_o e^{-\gamma x} = 0$$

$$\frac{\partial}{\partial x}\left(-\gamma V_o e^{-\gamma x}\right) - \gamma^2 V_o e^{-\gamma x} = 0$$

$$\gamma^2 V_o e^{-\gamma x} - \gamma^2 V_o e^{-\gamma x} = 0$$

$$0 = 0$$

Thus we validate the assumed solution. This tells us that in the frequency domain, the voltage or current on a transmission line decays exponentially:

$$V = V_O e^{-\gamma x}$$

where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = |\gamma|\angle\varphi = \alpha + j\beta$

γ = propagation constant

α = attenuation coefficient

β = phase coefficient

Where:

$$|\gamma| = \sqrt{\sqrt{R^2 + \omega^2 L^2}\sqrt{G^2 + \omega^2 C^2}} \qquad \varphi = \frac{1}{2}\left[\tan^{-1}\left(\frac{\omega L}{R}\right) + \tan^{-1}\left(\frac{\omega C}{G}\right)\right]$$

$$\alpha = \mathrm{Re}\left\{\sqrt{(R+j\omega L)(G+j\omega C)}\right\} = |\gamma|\cos(\varphi)$$

and:

$$\beta = \mathrm{Im}\left\{\sqrt{(R+j\omega L)(G+j\omega C)}\right\} = |\gamma|\sin(\varphi)$$

In exponential notation, a sinusoid may be represented by a rotating unity vector, of some frequency:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Note that the magnitude of this function is 1, but the phase angle is changing as a function of *t*.

If we let: $V_o = e^{j\omega t}$

Then

$$V = e^{j\omega t}e^{-\gamma x} = e^{j\omega t}e^{-(\alpha+j\beta)x} = \underbrace{e^{-\alpha x}}_{\substack{\text{attenuation} \\ \text{vs. } x}} \underbrace{e^{j(\omega t-\beta x)}}_{\substack{\text{phase} \\ \text{vs. } t \text{ and } x}}$$

This result is quite interesting because it is the same solution for the transmission line equation in the time domain. The term $e^{-\alpha x}$ represents an exponential decay. The signal is attenuated as length *x* increases. The amount of attenuation is defined as:

$$\text{Attenuation in Nepers} = \left|\ln e^{-\alpha x}\right| = \alpha x$$

$$\text{Attenuation in dB} = 20\log e^{-\alpha x} \approx 8.68589\alpha x$$

This allows us to determine the attenuation at any frequency at any point in a transmission line, if we are given the basic line parameters of *R*, *L*, *G*, & *C*.

The term $e^{j(\omega t-\beta x)}$ represents a rotating unity vector since:

$$e^{j(\omega t-\beta x)} = \cos(\omega t-\beta x) + j\sin(\omega t-\beta x)$$

The phase angle of this vector is βx radians.

Characteristic Impedance

The characteristic impedance of a transmission line is also known as its surge impedance, and should not be confused with its resistance. If a line is infinitely long, electrical signals will still propagate down it, even though the resistance approaches infinity. The characteristic impedance is determined from its AC attributes, not its DC ones.

Recall from our earlier analysis:

$$\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \quad \text{and} \quad \frac{\partial i}{\partial x} = Gv + C\frac{\partial v}{\partial t}$$

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Taking the Fourier Transform of these expressions, we obtain:

$$\frac{\partial}{\partial x} V = RI + j\omega LI \quad \text{and} \quad \frac{\partial}{\partial x} I = GV + j\omega CV$$

If voltage and current are forcing functions of frequency, they are not functions of distance. i.e. the frequency of a signal remains constant regardless of how long the transmission line is, therefor:

$$\frac{\partial}{\partial x} V = V \quad \text{and} \quad \frac{\partial}{\partial x} I = I$$

Consequently, we may write the above equations as:

$$V = RI + j\omega LI = (R + j\omega L)I$$
$$I = GV + j\omega CV = (G + j\omega C)V$$

Taking the ratio of these two attributes in order to obtain impedance:

$$\frac{V}{I} = \frac{(R + j\omega L)I}{(G + j\omega C)V}$$
$$\left(\frac{V}{I}\right)^2 = \frac{R + j\omega L}{G + j\omega C}$$
$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

This can be simplified for the low and high frequency case:

$$Z_{o(low\ freq)} = \sqrt{\frac{R}{G}}$$
$$Z_{o(high\ freq)} = \sqrt{\frac{L}{C}}$$

Skin Effect

Under DC conditions, electrons are uniformly spread throughout the cross-section of the conductor. However, as frequency increases, electrons have a tendency to redistribute themselves, and migrate towards the outer surface.

The skin depth is the distance from the surface, where the current density has dropped to 1/*e* of its surface value. [The conductor is assumed to have a thickness of at least three times the skin depth]. Under such conditions, the solid conductor can be replaced by a hollow one.

$$\text{skin depth} = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \text{ meters}$$

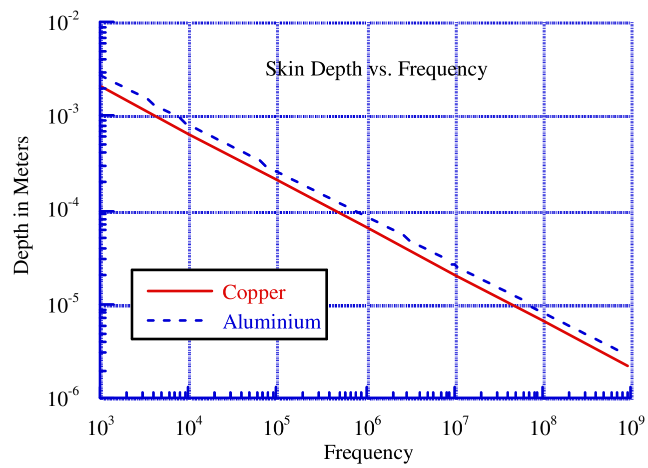
where

$\mu =$ permeability

$\sigma =$ conductivity

Conductivity of copper: $\sigma_{copper} = 5.81 \times 10^7$ mhos per meter.

Permeability of copper: $\mu_{copper} = 4\pi \times 10^{-7}$ henries per meter.



RELATIVE PERMEABILITY

Material	μ_r	Comments
Copper	0.99999	Diamagnetic
Silver	0.99998	"
Gold	0.99996	"
Bismuth	0.99983	"
Plastics	~1.0	"
Aluminum	1.000021	Paramagnetic
Titanium	1.00018	"
Palladium	1.00082	"
Nickel	250	Ferromagnetic
Cobalt	600	"

$\mu = \mu_r \mu_o$ where: $\mu_o = 4\pi \times 10^{-7}$ henries/meter

For most transmission line dielectrics: $\mu \approx \mu_o$

and since $\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}}$ then $\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_r \varepsilon_o}} \approx \frac{377}{\sqrt{\varepsilon_r}}$

RELATIVE DIELECTRIC CONSTANTS

Material	ε_r
Vacuum	1.0
Air	1.0006
Teflon	2.0
Polyethylene	2.25
Paraffin paper	2.5
Rubber	3.0
Mica	5.0
Glass	7.5

$$\varepsilon = \varepsilon_r \varepsilon_o \qquad \text{where:} \qquad \varepsilon_o \approx \frac{1}{36\pi} \times 10^{-9} \text{ farads/meter}$$

RELATIVE CONDUCTIVITY AND RESISTIVITY

Material	σ_r	ρ_r
Aluminum	0.610	1.64
Brass	0.256	3.9
Copper (annealed)	1.0	1.0
Gallium	0.0176	56.8
Gold	0.7062	1.416
Iron	0.178	5.6
Lead	0.0782	12.78
Mercury	0.01789	55.6
Nichrome I	0.01538	65.0
Nickel	0.198	5.05
Silver	1.05	0.95
Steel	0.0189	52.8
Tantalum	0.111	9.0
Tin	0.149	6.7
Titanium	0.0209	47.8
Tungsten	0.307	3.25
Zinc	0.294	3.4

$$\rho = \rho_r \rho_c \qquad \rho_c = 1.7241 \times 10^{-8} \text{ ohms/meter}$$

$$\sigma = \sigma_r \sigma_c \qquad \sigma_c = 5.81 \times 10^{-7} \text{ mhos/meter}$$

$$\sigma = \frac{1}{\rho}$$

$$R = \frac{\rho L}{A} \text{ ohms}$$

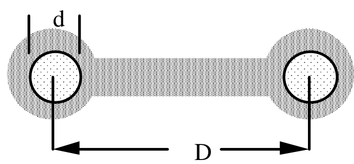
$L =$ length

$A =$ cross-sectional area

$$\text{resistance per square} = R_{sq} = \frac{1}{\delta\sigma} \text{ ohms}$$

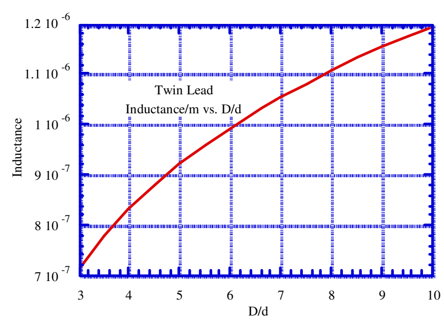
Twin Lead Cable

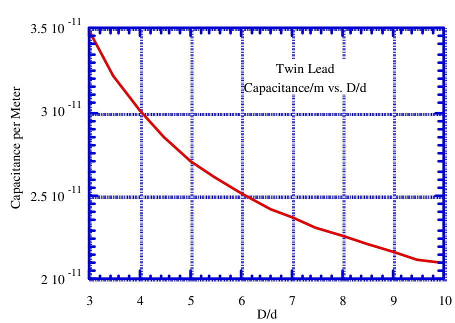
Twin lead cables are used to connect television receivers to set or roof mounted antennas. The typical characteristic impedance of these cables is 300Ω.



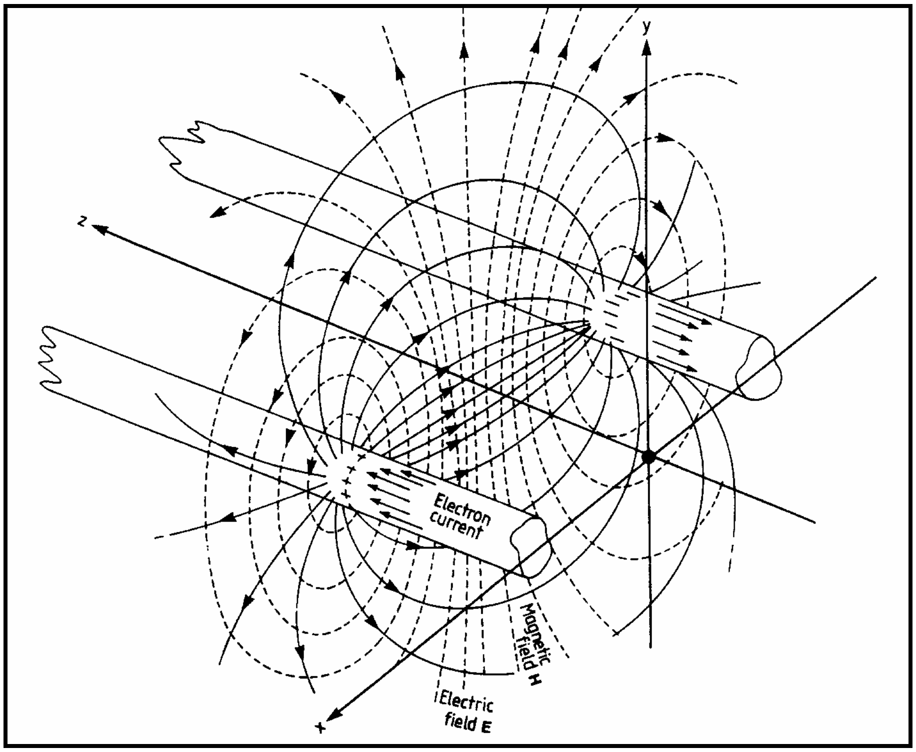
for: $$\left(\frac{D}{d}\right)^2 >> 1$$

$$L \approx \frac{\mu}{\pi}\ln\left(\frac{2D}{d}\right) \quad \text{h/m} \qquad C \approx \frac{\pi\varepsilon}{\ln\left(\frac{2D}{d}\right)} \quad \text{f/m}$$



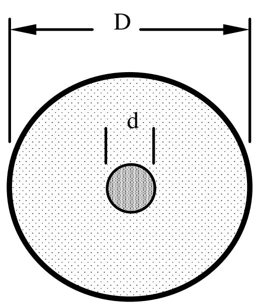


TWIN LEAD FIELDS



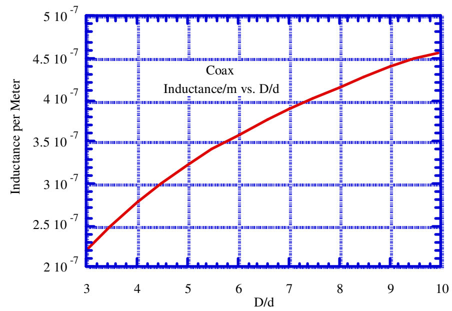
Coaxial Cable

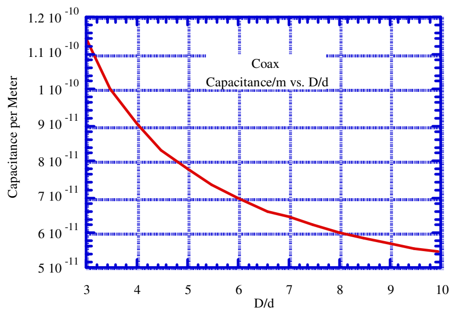
Coax is used primarily in cable TV applications. The typical characteristic impedance of these cables is 75Ω.



for: $\left(\frac{D}{d}\right)^2 >> 1$

$$L \approx \frac{\mu}{2\pi}\ln\left(\frac{D}{d}\right) \text{ h/m} \qquad C \approx \frac{2\pi\varepsilon}{\ln\left(\frac{D}{d}\right)} \text{ f/m} \qquad G \approx \frac{2\pi\sigma}{\ln\frac{D}{d}} \text{ mhos/m}$$





Transient Analysis

When a DC source is attached to a transmission line, a voltage pulse travels down the line the far end towards the load. If the voltage surge meets any discontinuity or change in impedance, a portion of the signal will be reflected back to the source. Equilibrium will eventually be achieved as the resulting bouncing signals diminish to zero.

Assuming a lossless transmission line, the fraction of the voltage reflected back to its origin, is known as the reflection coefficient, and is given by:

$$\Gamma_D = \frac{Z_D - Z_o}{Z_D + Z_o}$$

where

$\Gamma_D =$ reflection at the discontinuity

$Z_D =$ impedance at the discontinuity

$Z_o =$ characteristic impedance of the transmission line

The instantaneous voltage on a transmission line can be thought of as being composed of three components:

- The initial condition
- The incoming signal
- The outgoing reflection

EXAMPLE

In the circuit illustrated below, let:

V_S = 10 volts

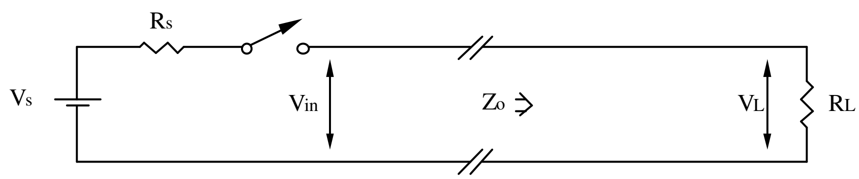
R_S = source resistance of 150 Ω

Z_o = transmission line characteristic impedance of 50 Ω

Z_L = load impedance of 25 Ω

V_S = source voltage of 10 volts

T = propagation time down the transmission line



If the switch is closed at t = 0, the instantaneous voltage V_{in} is given by:

$$V_{in}\Big|_{t=0} = V_S \times \frac{Z_o}{R_S + Z_o} = 10 \times \frac{50}{150+50} = 2.5v$$

The reflection coefficient at the load is given by:

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{25-50}{25+50} = -\frac{1}{3}$$

The reflection coefficient at the source is given by:

$$\Gamma_S = \frac{Z_S - Z_o}{Z_S + Z_o} = \frac{150-50}{150+50} = \frac{1}{2}$$

Since it takes T seconds for the input signal to travel down the line, the initial voltage at V_L = 0.

When *t* = T, the input signal arrives and a voltage appears at V_L. and the total result is:

$$V_L\Big|_{t=T} = \underbrace{V_r}_{\text{initial condition}} + \underbrace{V_{in}}_{\text{incoming signal}} + \underbrace{V_{in} \times \Gamma_L}_{\text{reflected signal}} = 0 + 2.5 + \underbrace{2.5 \times \frac{-1}{3}}_{-.83333} = 1.6667v$$

Notice that in this example, the reflected signal is negative and subtracts from the incoming signal. Had R_L been greater than Z_o, the reflected signal would have been positive. The -0.8333 v reflected signal component is sent back to the source where it is reflected again. At *t = 2T*, the voltage at V_{in} is given by:

$$V_L\big|_{t=2T} = \underbrace{V_r}_{\text{initial condition}} + \underbrace{V_{in}}_{\text{incoming signal}} + \underbrace{V_{in} \times \Gamma_S}_{\text{reflected signal}} = 2.5 - .8333 - \underbrace{.8333 \times \frac{1}{2}}_{-.4167} = 1.2501v$$

The -0.4167 volt reflection is sent back down to the load:

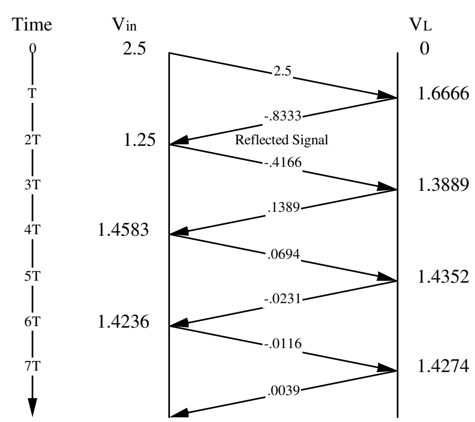
$$V_L\big|_{t=3T} = 1.6667 - .4167 - .4167 \times \frac{-1}{3} = 1.3889v$$

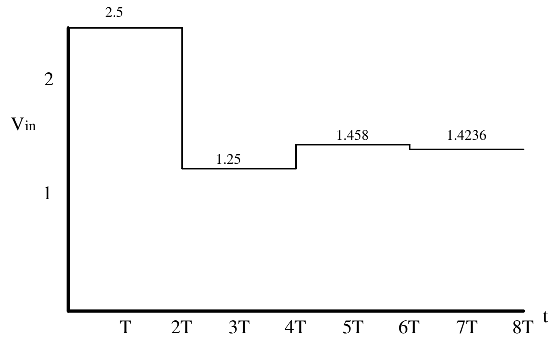
This process continues until equilibrium is reached. Both ends of this lossless loop will converge at a final value of:

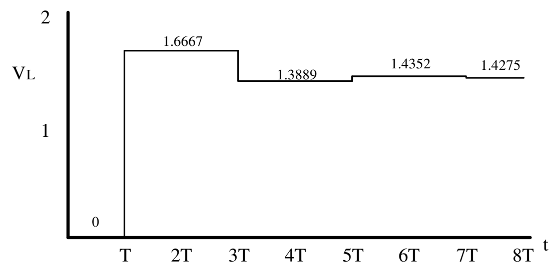
$$V_{final} = V_{in} = V_L = V_S \times \frac{R_L}{R_S + R_L} = 10 \times \frac{25}{150 + 25} = 1.429v$$

Time	V_{in}	V_L
0	2.5	0
T	2.5	1.6667
2T	1.25	1.6667
3T	1.25	1.3889
4T	1.4583	1.3889
5T	1.4583	1.4352
6T	1.4236	1.4352
7T	1.4236	1.4275

The derivation of these numbers is often more readily apparent from a ‘bounce diagram’. This type of sketch shows the size of the signal that is reflected, back and forth between the two ends of the transmission line.







From the forgoing analysis, we can conclude that:

Termination	Voltage Reflection	Current Reflection
Open circuit	Total, in phase, positive	Total, out of phase, negative
$Z_L > Z_O$	Partial, in phase positive	Partial, out of phase, negative
$Z_L = Z_O$	None	None
$Z_L < Z_O$	Partial, out of phase, negative	Partial, in phase positive
Short circuit	Total, out of phase, negative	Total, in phase, positive

Reflections are undesirable. They send power back to the source, which may ultimately cause damage, and they can lead to the development of standing waves when the transmission line is continuously excited.

Standing Waves

If waves are allowed to reflect back and forth on a transmission line, the incident and reflected waves will interact to create a standing wave. This is somewhat similar to the vibration of a stringed instrument. The ratio of maximum to minimum voltage of a standing wave is known as the VSWR† .

$$VSWR = \frac{E_{\max}}{E_{\min}} = \frac{I_{\max}}{I_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

The VSWR can take on any value between 1 and ∞.

† Voltage Standing Wave Ratio

If the load is purely resistive, this expression may be simplified to:

$$VSWR = \underbrace{\frac{Z_o}{R_L} \text{ or } \frac{R_L}{Z_L}}_{\text{whichever is larger}}$$

If VSWR = ∞, total reflection occurs. Ideally for a matched load, VSWR = 1 and there are no reflections. The energy contained in the reflected wave may be dissipated in the cable itself as I^2R losses or absorbed by the generator.

A VSWR ≠ 1may be acceptable if the transmission line is used as a tuned circuit or a reactive load.

Quarter Wavelength Impedance Transformer

To terminate a transmission line with a resistive load R_L not equal to its characteristic impedance, a λ/4 section with a characteristic impedance of $Z' = \sqrt{Z_o R_L}$ can be placed between the two as an impedance transformer.

Two reflections occur; one at the input of the matching section, and one at the load, but they are equal and anti-phase and therefore cancel out.

The impedance of any point along a transmission line is given by:

$$Z = Z_o \left[\frac{Z_L + jZ_o \tanh \gamma x}{Z_o + jZ_L \tanh \gamma x} \right]$$

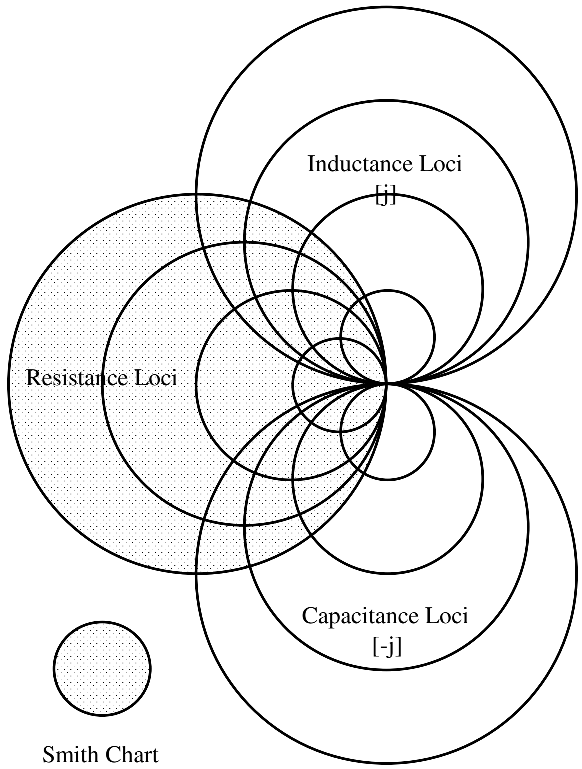
This expression simplifies to $Z = Z_o \left[\frac{Z_L + jZ_o \tan \beta x}{Z_o + jZ_L \tan \beta x} \right]$ for the lossless transmission line [$R = G = 0$].

Smith Chart

All values on the Smith chart are normalized to the characteristic impedance of the line.

$$Z' = \frac{Z}{Z_o}$$

The location of all impedances along the transmission line are described by circles on a Smith chart.


Inductance Loci
[j]
Resistance Loci
Capacitance Loci
[-j]
Smith Chart

Assignment Questions

Quick Quiz

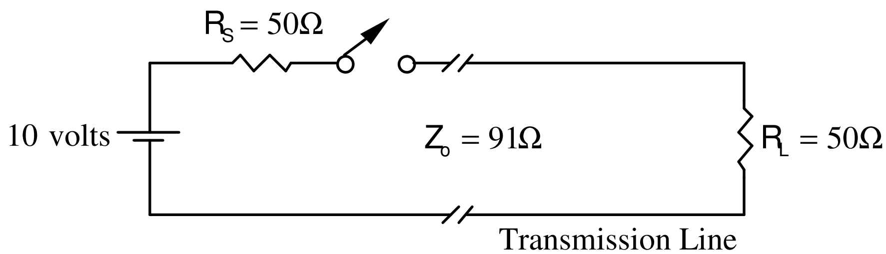
1. Signals are attenuated [linearly, exponentially] as they propagate down a transmission line.

Analytical Questions

5. Show that the attenuation coefficient for a distortionless transmission line is given by:

$$\alpha \approx \sqrt{RG}$$

5. Given the following circuit:



a) Calculate the reflection coefficient

b) Sketch the voltage waveform going into the transmission line and at the load for 5 propagation time constants, when the switch is closed

6. A coaxial cable with a .75 mm center conductor diameter, using Teflon as the dielectric (ε_r = 2), has a characteristic impedance of 75 Ω. Stating all assumptions, find:

a) The diameter of the outer conductor

b) The inductance per meter

c) The capacitance per meter

d) The phase velocity

7. A 150 MHz, 1 Vrms signal is injected into a transmission line with the following characteristic values:

R = .025 Ω/m

L = .02 nH/m

C = .15 pfd/m

G = 1 μS/m

a) Find the signal amplitude and phase angle after 1000 meters

b) Calculate the low and high characteristic impedance

c) Find the magnitude and phase angle of the cable impedance at 250 MHz

d) Find the magnitude of the impedance half way along the 1 Km cable, if it is terminated in 50 Ω.

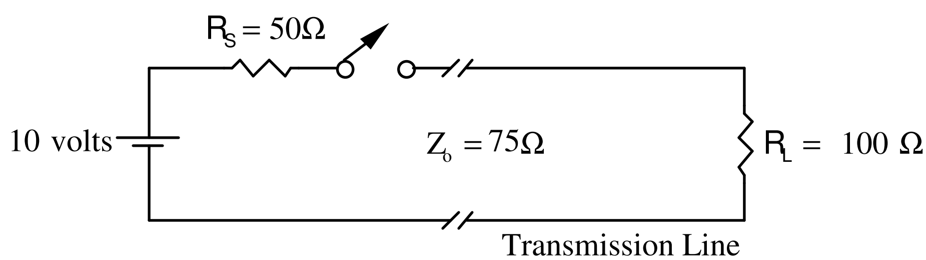
8. Explain why a lumped model of a transmission line does not give the same results as a distributed model.

9. What is another name for characteristic impedance?

10. Fill in the reflection polarity in the following table:

	Voltage Reflection	Current Reflection
Open Circuit		
Short Circuit		

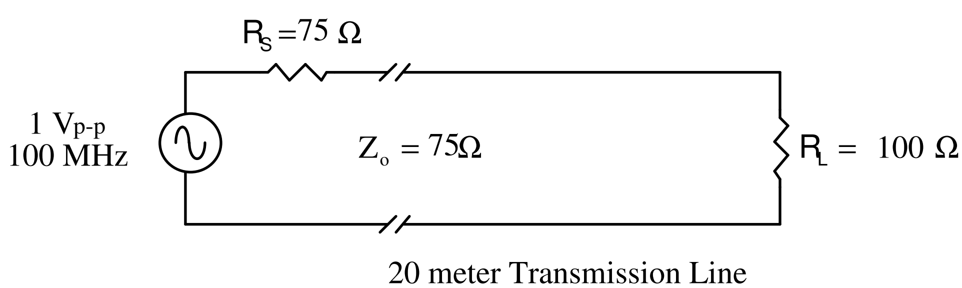
11. Given the following circuit:



a) Sketch the approximate waveforms going into the transmission line and at the load for 3 propagation time constants, when the switch is closed.

b) Find the final quiescent voltage on the transmission line.

c) Calculate the VSWR if the circuit is modified as follows:



12. A coaxial cable with a 2 mm center conductor, uses Teflon as the dielectric, and has a characteristic impedance of 91 Ω. Stating all assumptions, find:

a) the diameter of the outer conductor

b) The inductance per meter

c) The capacitance per meter

Composition Questions

1. What is a lossless transmission line?
2. Sketch and label the field patterns for a twin lead cable.
3. What conditions must be met for distortionless transmission?

For Further Research

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